Maxwell Equations

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Syllabus

- **Electrodynamics**: Maxwell's equations: differential and integral forms, significance of Maxwell's equations, displacement current and correction in Ampere's law, electromagnetic wave propagation, transverse nature of EM waves, wave propagation in bounded system, applications.

- **Quantum Physics** Dual nature of matter, de-Broglie Hypothesis, Heisenberg uncertainty principle and its applications, postulates of quantum mechanics, wave function & its physical significance, probability density, Schrodinger's wave equation, Eigen values & Eigen functions, Applications of Schrodinger equation.

Prerequisite… E & M

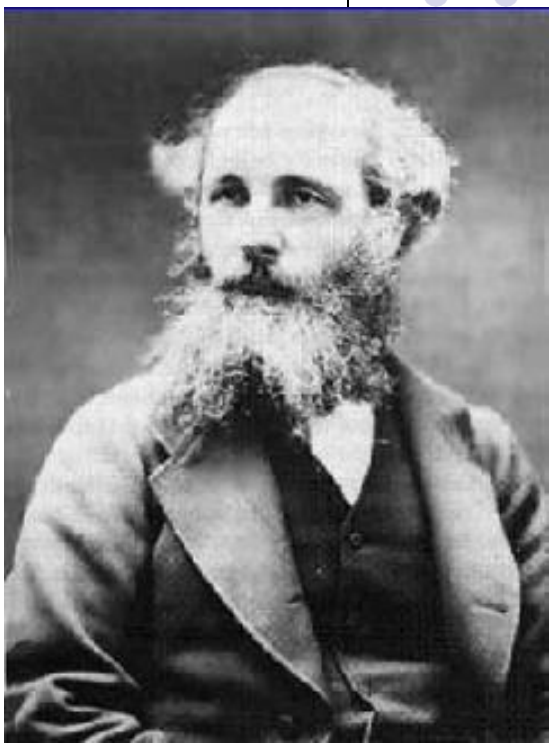
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The History of Electromagnetics

The history of electromagnetics shows that it is a series of discoveries of many people instead of by just one person.

James Clerk Maxwell (Dafty)

- Born into a wealthy family in Edinburgh. Was well educated and inquisitive.
- Went to Cambridge college, but moved to Trinity for competitive reasons.
- Was a professor at Trinity, Aberdeen, and London.
- Put Faraday's Law in mathematical form.
- Discovered the famous 4 equations that govern electromagnetics.
- Devout Christian.



Quantities

- In mathematics there are two types of quantities:
  - Vector: Direction + Magnitude
  - Scalar: Only Magnitude
- Gradient: An operator in vector calculus.

The Del Operator

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

The Gradient

The Definition of the Gradient

The *gradient* of a scalar field is defined as:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{4.6.1}$$

Using the gradient, we can define the directional derivative of f in the direction of a unit vector $\hat{u}$ more concisely.

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} \tag{4.6.2}$$

Gradient takes in a ***scalar function*** and outputs ***vector field***

Points in the ***direction of the steepest increase***

Two key concepts in vector calculus are **divergence** and **curl,** the latter of which is sometimes called *circulation.* Basically, **divergence** has to do with how a vector field changes its magnitude in the neighborhood of a point, and **curl** has to do with how its direction changes.

Definition Of Divergence

Let $\mathbf{v}(x, y, z)$ be a differentiable vector function,

$$\mathbf{v}(x, y, z) = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$$

Then the function

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the **divergence** of $\mathbf{v}$ or the *divergence of the vector field defined by* $\mathbf{v}$.

Another convenient notation for div is as follows:

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) =$$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

"Del" operator $\vec{\nabla} \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)$

gradient $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

The Formulas for the Divergence and Curl

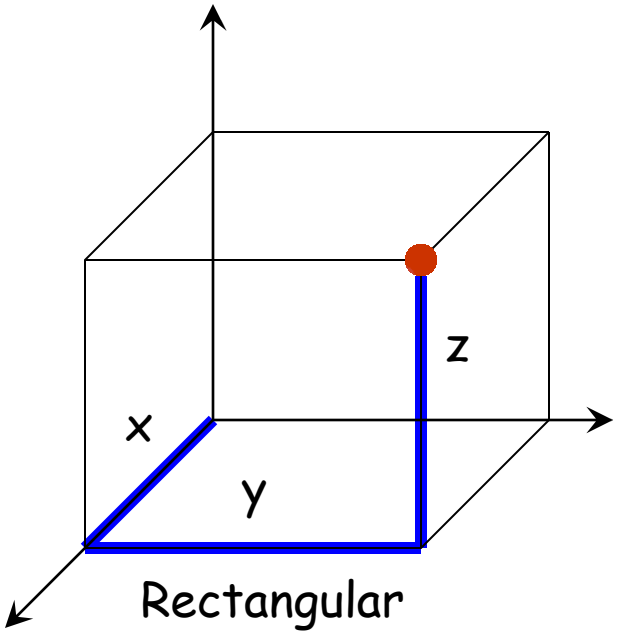
Given a vector field $\vec{f} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$ the divergence (written $\vec{\nabla} \cdot \vec{f}$) and curl (written $\vec{\nabla} \times \vec{f}$) are given by:

divergence
$$\vec{\nabla} \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \qquad (8.6.1)$$

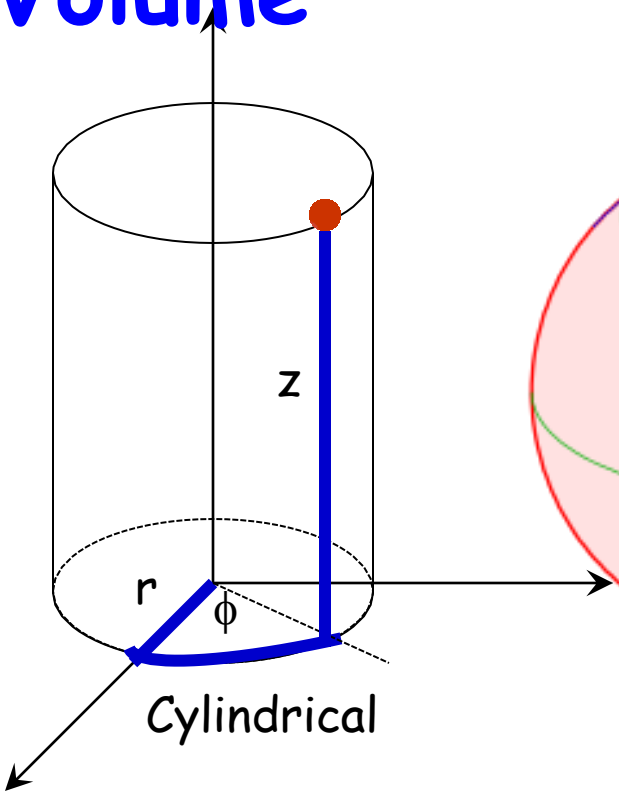
curl
$$\vec{\nabla} \times \vec{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right)\hat{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right)\hat{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right)\hat{k} \qquad (8.6.2)$$

curl
$$\vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

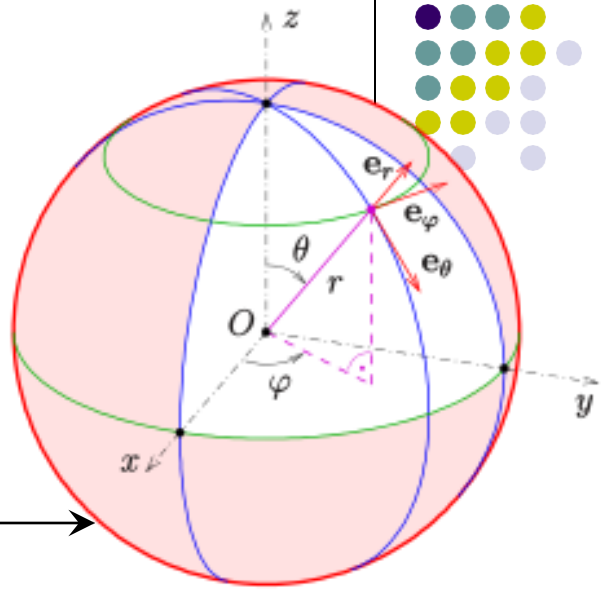
Length, Area, Volume



Rectangular



Cylindrical



Spherical

$dl^2 = dx^2 + dy^2 + dz^2$

$dA_z = \pm dx.dy$
$dA_x = \pm dy.dz$
$dA_y = \pm dz.dx$

} 3 sets of surfaces

$dV = dx.dy.dz$

$dl^2 = dr^2 + r^2d\phi^2 + dz^2$

$dA_z = \pm dr.(rd\phi)$
$dA_r = \pm dz.(rd\phi)$
$dA_\phi = \pm dz.dr$

$dV = rdr.d\phi.dz$

$dl^2 = dR^2 + R^2d\phi^2 + R^2\sin^2\theta dz^2$

$dA_r = \pm(Rd\theta).(R\sin\theta d\phi)$
$dA_\theta = \pm dR.(R\sin\theta d\phi)$
$dA_\phi = \pm dR.Rd\theta$

$dV = R^2\sin\theta dR.d\theta.d\phi$

Materials

- ✓ Comprises of many atoms.
- ✓ Atoms have neutron, proton and electrons.
- ✓ Protons are positively charged.
- ✓ Electrons are negatively charged.
- ✓ Interactions among the particles through long range as well as short range forces.

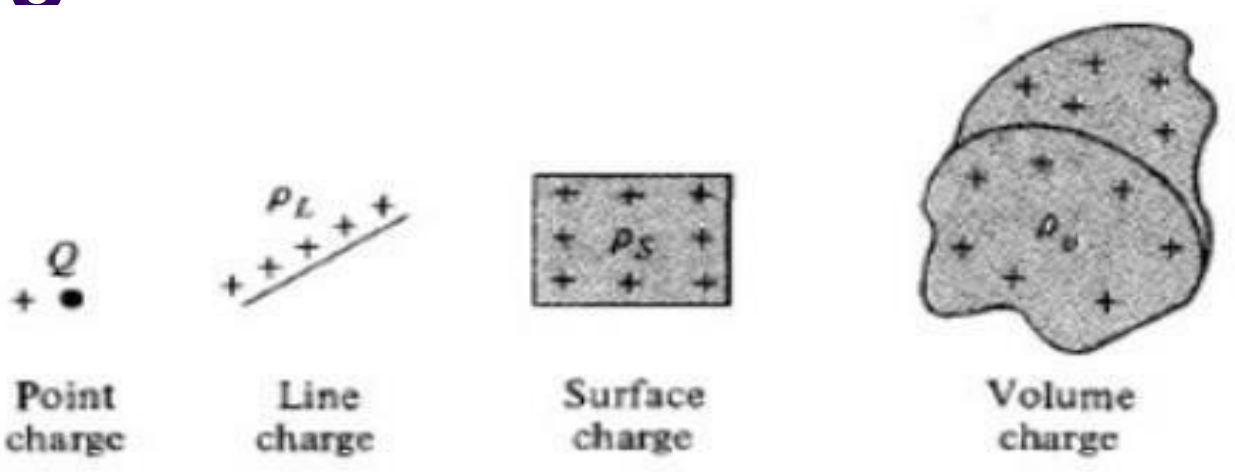
Electrons

Are located on the outer edges of atoms…they can be moved.

A concentration of electrons in an atom creates a net negative charge.

If electrons are stripped away, the atom becomes positively charged.

Charge distribution



- Total charge due to these distributions are given by:

$$Q = \int_L \rho_L \, dl \qquad \text{(line charge)}$$

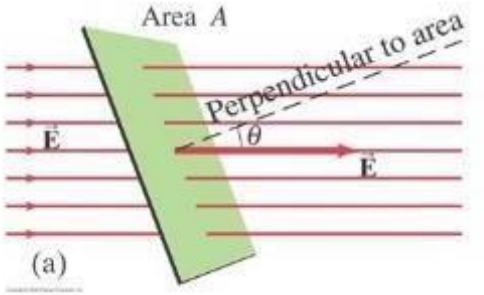
$$Q = \int_S \rho_S \, dS \qquad \text{(surface charge)}$$

$$Q = \int_v \rho_v \, dv \qquad \text{(volume charge)}$$

$$\mathrm{E} = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o |\mathbf{r} - \mathbf{r}'|^3}$$

Electric flux

- The electric flux refers to the electric field lines that penetrate a given surface

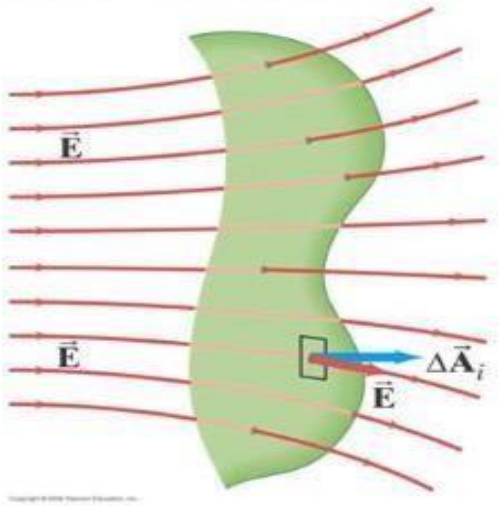


- For a given uniform electric field *E* passing thru an area *A*, the electric flux is defined as:

$$\Phi_E = EA\cos\theta$$

Flux For a Closed Area

- In general, an electric field is *not* uniform and the area is *not* flat !



- Any area can be divided up into *n* number of small elements of surface area ΔA, sufficiently small to be considered flat and the E-field is rather invariant over this small area.

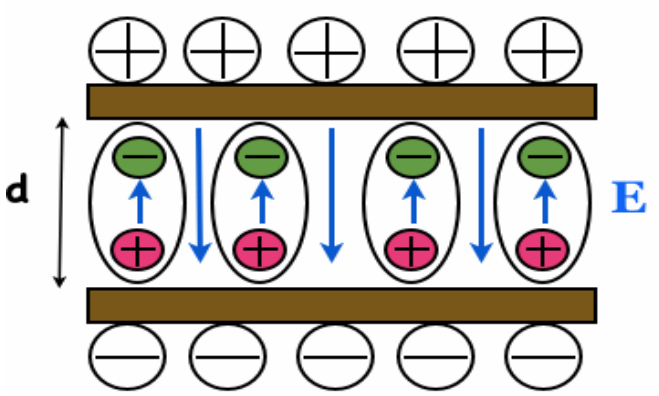
- Then flux is given by: $\Phi_E = \oint \vec{E} \bullet dA$

Permittivity	Permeability
A measure of the opposition offered by any material for against the formation of an electric field	It is the measure of the extent to which magnetic field lines can enter substance.
$Permittivity = \frac{Electric\ displacement}{Electric\ field\ intensity}$	$Permeability = \frac{Magnitude\ of\ magnetic\ induction(B)}{Intensity\ of\ magnetic\ field(H)}$
ϵ	μ
Farad/metre or F/m	Henries/metre or H/m
The permittivity of Free Space $\epsilon_0 = 8.85 \times 10^{-12}$ Farad/metre	The permeability of free space $\mu_0 = 4\pi \times 10^{-7}$ H/m
Electric field	Magnetic field
Polarisation of electric charges	Magnetization
Capacitors	Transformer core and Inductors

Permittivity

- Measure of a material's ability to **resist** an electric field.
- Denoted by the symbol .

$$\varepsilon = \varepsilon_r \varepsilon_0$$



Gauss law of electric fields

- The total of the electric flux out of a closed surface is equal to the charge enclosed divided by the permittivity.

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},$$

The Need for Poisson's and Laplace's Equations (Cont'd)

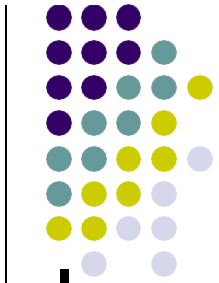


Poisson's equation is a differential equation for the electrostatic potential *V*. Poisson's equation and the boundary conditions applicable to the particular geometry form a boundary-value problem that can be solved either analytically for some geometries or numerically for any geometry.

After the electrostatic potential is evaluated, the electric field is obtained using

$$\underline{E}(\underline{r}) = -\nabla V(\underline{r})$$

Derivation of Poisson's Equation



For now, we shall assume the only materials present are free space and conductors on which the electrostatic potential is specified. However, Poisson's equation can be generalized for other materials (dielectric and magnetic).

Derivation of Poisson's Equation (Cont'd)

$$\nabla \cdot \underline{D} = q_{ev} \rightarrow \nabla \cdot \underline{E} = \frac{q_{ev}}{\varepsilon_0}$$

$$\underline{E} = -\nabla V \rightarrow \nabla \cdot \nabla V = -\frac{q_{ev}}{\varepsilon_0}$$

$$\nabla^2 V$$

Derivation of Poisson's Equation (Cont'd)

$$\nabla^2 V = -\frac{q_{ev}}{\varepsilon_0}$$

Poisson's equation

$\Rightarrow$ ∇^2 **is the *Laplacian operator*. The *Laplacian* of a scalar function is a scalar function equal to the divergence of the gradient of the original scalar function.**

Laplacian Operator in Cartesian, Cylindrical, and Spherical Coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Laplace's Equation



Laplace's equation is the homogeneous form of ***Poisson's equation***.

We use Laplace's equation to solve problems where potentials are specified on conducting bodies, but no charge exists in the free space region.

$$\nabla^2 V = 0$$

Laplace's equation

1. You should use Poisson's equation when your solution region contains space charges, and if you do not have space charges(practically, it is impossible), you can use The Laplace equation.
2. Poisson's equation takes care of volume charge density while Laplace's equation does not.
3. Solving the Poisson equation amounts to finding the electric potential φ for a given charge distribution. The mathematical details behind Poisson's equation in electrostatics are as follows (SI units are used rather than Gaussian units, which are also frequently used in electromagnetism).
4. Laplace's equation is a second-order partial differential equation widely useful in physics because its solutions R (known as harmonic functions) occur in problems of electrical, magnetic, and gravitational potentials, steady-state temperatures, and of hydrodynamics.

Magnetism

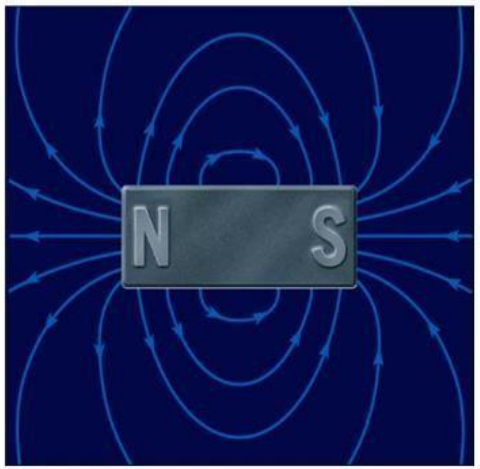
Is a physical phenomenon produced by the motion of electric charge, which results in attractive and repulsive forces between objects.

Magnetic Field

Is a region around a magnetic material or a moving electric charge within which the force of magnetism acts. Magnet produces magnetic force and have magnetic field lines.

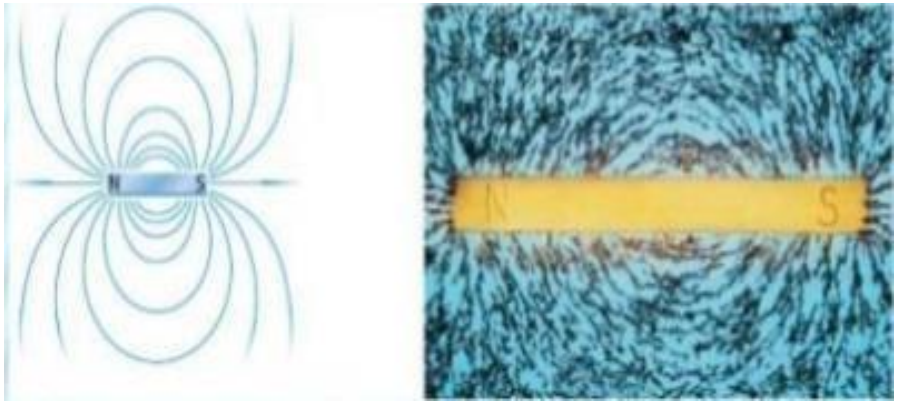
Magnetic Field

- **Definition**: Surrounds a magnet and exerts a force on other magnets
- **Characteristic**: A magnetic field is strongest close to the magnetic poles and weaker as you move further from the poles.
- Magnetic fields can be produced by moving electric charges.



Magnets

- Magnets have two poles.
  - North pole
  - South pole
- Opposite pole attract each other
- Similar pole repel each other.



Permeability

- Is the degree of magnetization that a material obtains in response to an applied magnetic field
- Tells, how easily an external magnetic field can induce an internal field in the material

$$B = \mu H$$

- Where, B = induced magnetic field
- H = externally applied magnetic field

Gauss law of Magnetism

- The net **magnetic** flux out of any closed surface is zero.
- For any closed surface ,the **magnetic** flux directed inward toward the south pole will equal the flux outward from the north pole.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

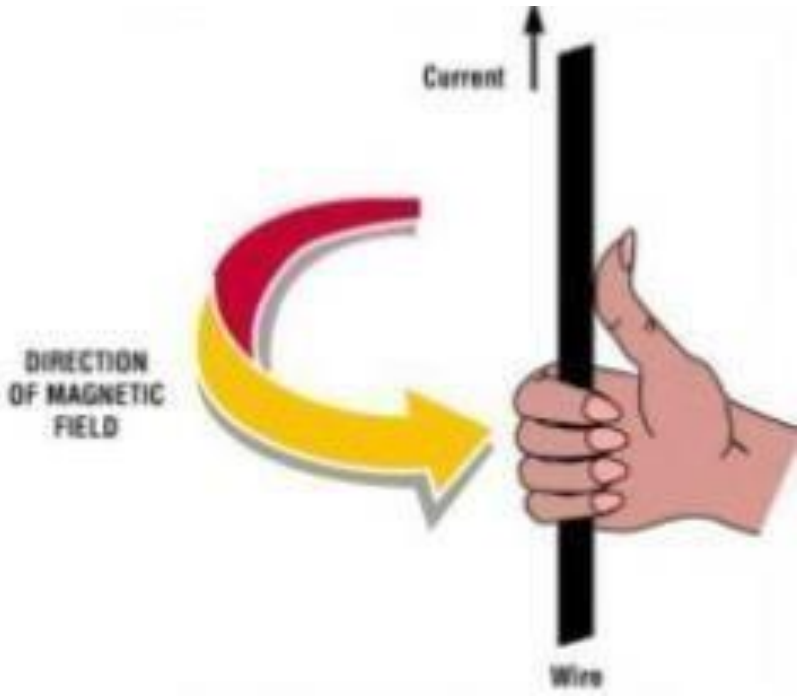
Electromagnetism

- Moving charge create a magnetic field in the direction perpendicular to the current.

- Direction of magnetic field is given by right hand rule

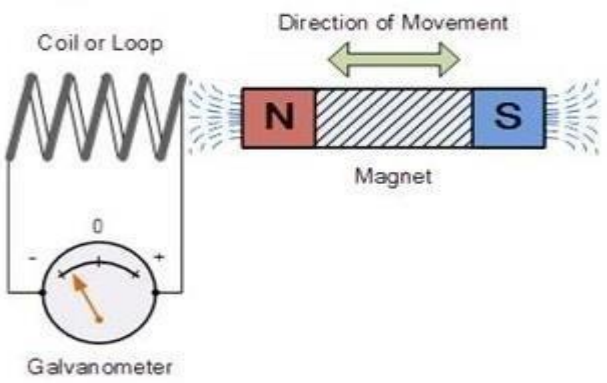
  .

  - Thumb- direction of current
  - Fingers – direction of magnetic field.



Faraday's Law

Any change in the magnetic field of a coil of wire will cause an EMF to be induced in the coil.
This EMF induced is called induced EMF and if the conductor circuit is closed, the current will also circulate through the circuit and this current is called induced current.



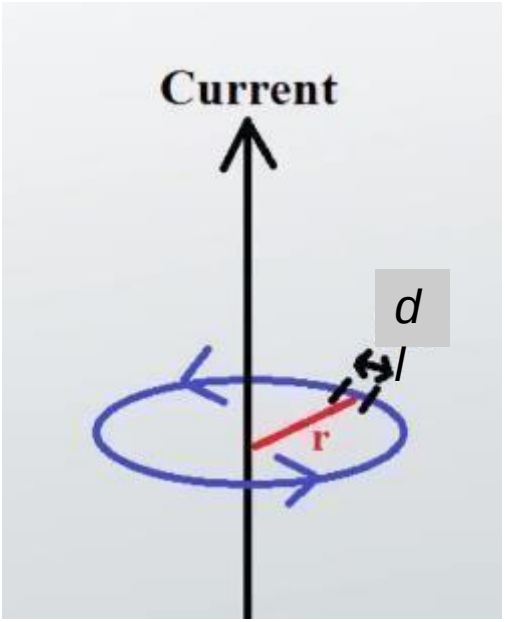
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

Ampere Law

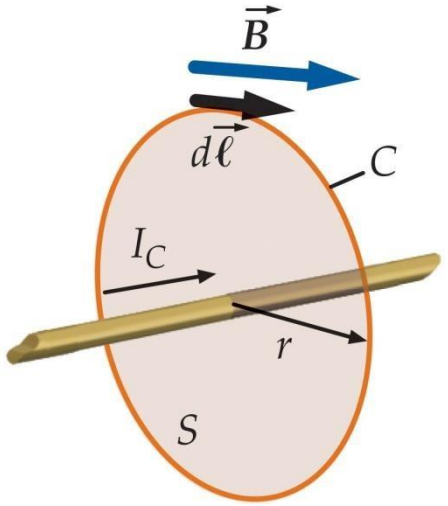
Magnetic field created by an electric current is proportional to the current with constant of proportionality equal to the permeability of free space.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$



Ampère's law

We found for highly symmetric charge distributions, we could calculate the electric field more easily using Gauss's Law than Coulomb's Law.
Similarly, Ampère's Law relates



the tangential component B_t of the magnetic field summed (integrated) around a closed curve C

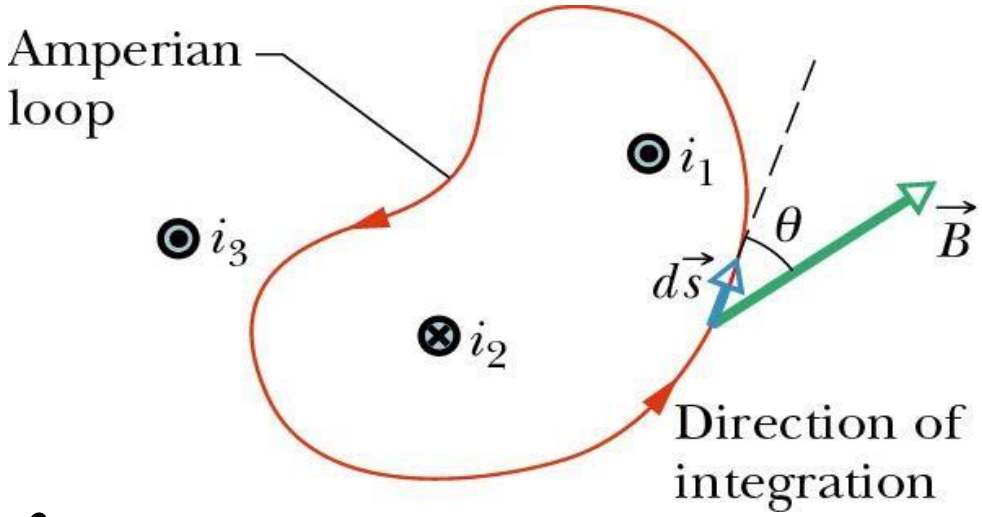
to

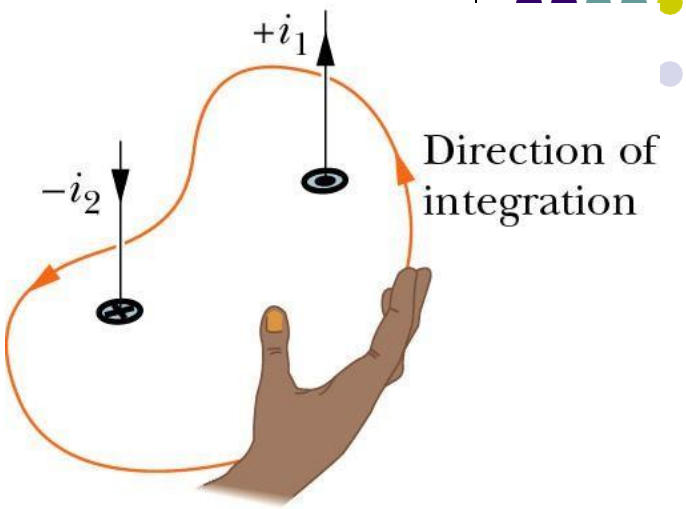
the current I_C that passes through any surface bounded by C.

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{encircled}} \quad \text{(Ampère's law, steady currents)} \qquad (26.16)$$

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$I_{\text{encircled}}$ is the net current that penetrates the surface bounded by the curve C.





$$\oint_C \underline{B} \cdot d\underline{s} = B\cos\theta\, ds = \mu_0 I_{\text{enclosed}}$$

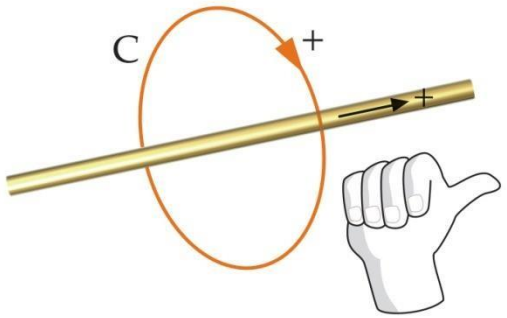
As in Gauss's law, the field that appears in the integral is the *net* field arising from *all* sources, not just the encircled currents.

We can now use Ampère's Law for situations where symmetry allows us to simplify, and solve the integral for the magnetic field.

$$\oint_C \underline{B} \cdot \underline{dl} = B\oint dl$$

The simplest case is for an infinitely long, straight, current-carrying wire.

Choose the circular curve centered on the wire. The magnetic field is tangent to this circle from the Biot-Savart law and has the same magnitude around the loop.



$$\oint_C \underline{B} \cdot d\underline{l} = B \oint dl = B 2\pi R = \mu_0 i_{\text{enclosed}}$$

$$B = \frac{\mu_0 i_{\text{enclosed}}}{2\pi R}$$

Maxwell Equations

Differential form of Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Integral form of Maxwell's equations

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{enc}$$

Contents:

- ✓ Maxwell's Equations
- ✓ Different media for propagation of EM waves
- ✓ Wave Equations in a DIELECTRIC medium

Maxwell's Equations:

- ✓ Light is an electromagnetic (EM) wave
- ✓ EM waves are described by Maxwell's equations
- ✓ These equations are based on experimental laws

Maxwell's Equations:

Maxwell's equations:

4 equations in general form

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's Equations:

Maxwell's equations:

4 equations in general form

$$\nabla \cdot \vec{D} = \rho$$

electric field from a charge distribution: Gauss' law

$$\nabla \cdot \vec{B} = 0$$

no magnetic monopoles

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

time varying magnetic field induces an electric field : electromagnetic induction

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

currents and time-varying electric fields produce magnetic fields

Maxwell's Equations:

Maxwell's equations:

4 equations in general form

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$\rho \rightarrow$ the charge density ; $\vec{J}$, the current density

$\vec{E} \rightarrow$ electric field; $\vec{H}$, the magnetic field

$\vec{D} \rightarrow$ electric displacement ; $\vec{B}$, magnetic induction

Constitutive relations

$\vec{D} = \tilde{\varepsilon}\vec{E}$ $\quad$ $\vec{B} = \tilde{\mu}\vec{H}$ $\quad$ $\vec{J} = \sigma\vec{E}$

$\tilde{\varepsilon} \rightarrow$ dielectric permittivity

$\tilde{\mu} \rightarrow$ magnetic permeability

$\sigma \rightarrow$ conductivity of the medium

It may be noted :

- ✓ Except the term for displacement current $\frac{\partial \vec{D}}{\partial t}$ all the experimental laws given in these equations were known before Maxwell

- ✓ Introducing this concept Maxwell could derive wave equation and predict the existence of EM waves

EM waves in different media: Maxwell's Equations

Propagation of EM waves: the medium could be

- ✓ Isotropic or anisotropic,
- ✓ homogeneous or inhomogeneous,
- ✓ linear or nonlinear

EM waves in different media: Maxwell's Equations

Propagation of EM waves: the medium could be

- ✓ dielectric medium
- ✓ Interfaces and layers
- ✓ absorbing medium

Maxwell's Equations:

Therefore, Maxwell's equations in a linear isotropic and homogeneous dielectric

$$\nabla \cdot \vec{E} = 0 \qquad (1a)$$

$$\nabla \cdot \vec{H} = 0 \qquad (1b)$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (1c)$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (1d)$$

Maxwell's Equations:

Taking curl of equation $(1c)$

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$ using curl equation $(1d)$

For the LHS use the identity

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

Maxwell's Equations:

Since $\nabla \cdot \vec{E} = 0$: equation $(1b) \Rightarrow \nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

In a similar way, $\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$

taking curl of equation $(1d)$ and using curl equation $(1c)$

Linear Isotropic Homogeneous Dielectric

The pair of wave equations for the $\vec{E}$ and $\vec{H}$ fields

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

- ✓ These equations are the most general form
- ✓ EM waves in linear isotropic and homogeneous dielectric

EM waves in perfect dielectric:

The pair of wave equations for the $\vec{E}$ and $\vec{H}$ fields

For a perfect dielectric conductivity $\sigma = 0$

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \Longrightarrow \nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \Longrightarrow \nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

Linear Isotropic Homogeneous Dielectric

so for a perfect dielectric, $\sigma = 0$

the wave equations for $\vec{E}$ and $\vec{H}$ fields

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

- ✓ These are vector wave equations
- ✓ Any component of each equation gives scalar wave equation

EM waves in perfect dielectric:

- ✓ so the general form of vector wave equations for the $\vec{E}$ and $\vec{H}$ fields

Cartesian components → E_X, E_Y, E_Z and H_X, H_Y, H_Z

spherical coordinates → E_r, E_θ, E_ϕ and H_r, H_θ, H_ϕ

cylindrical coordinates → E_ρ, E_ϕ, E_Z and H_ρ, H_ϕ, H_Z

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

- ✓ if we consider any of the 3 components of $\vec{E}$ or $\vec{H}$ fields in the respective equations,
- ✓ we will obtain a scalar wave equation

EM waves in perfect dielectric:

$$\nabla^2\psi = \mu\varepsilon\frac{\partial^2\psi}{\partial t^2}$$

The scalar wave equations for both the $\vec{E}$ and $\vec{H}$ fields

where ψ represents any of the following field components

Cartesian components → E_X, E_Y, E_Z and H_X, H_Y, H_Z

spherical coordinates → E_r, E_θ, E_ϕ and H_r, H_θ, H_ϕ

cylindrical coordinates → E_ρ, E_ϕ, E_Z and H_ρ, H_ϕ, H_Z

Existence EM Waves: Speed Estimation

comparing these equations with the form of wave equation

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \Rightarrow v = \frac{1}{\sqrt{\mu\varepsilon}}$$

- ✓ Maxwell's equations predict EXISTENCE of EM waves
- ✓ Estimates the SPEED of EM waves in such a medium

Experimental Verification

for EM waves in FREE SPACE

using $\varepsilon = \varepsilon_0 \approx 8.854 \times 10^{-12}\, C^2/Nm^2$
and $\mu = \mu_0 \approx 4\pi \times 10^{-7}\, N/A^2$

SPEED of EM waves predicted

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$= \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 2.99794 \times 10^8\, m/s$$

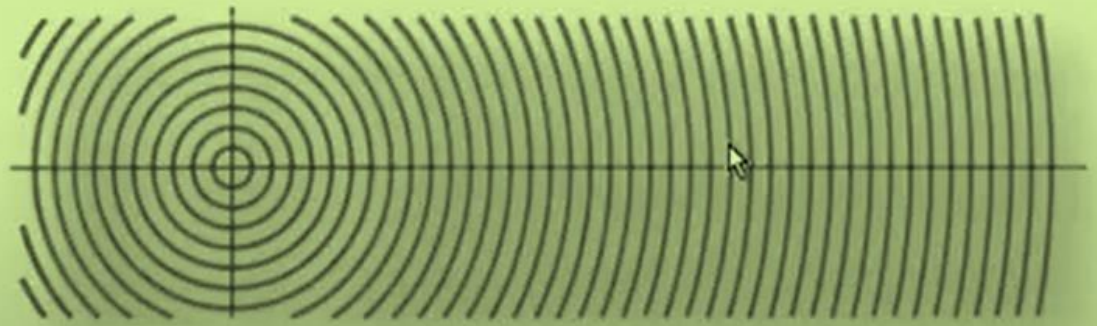
⇦ several other measurements by Kohlrausch, Weber agreed so well

SPEED measured by FIZEAU in 1849

⇨ $v = 3.14858 \times 10^8\, m/s$

Contents:

- ✓ Solution in Cartesian coordinates : plane waves
- ✓ Solution in spherical coordinates : spherical waves
- ✓ Solution in cylindrical coordinates : cylindrical waves



Electromagnetic Waves:

The pair of wave equations:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$ for $\vec{E}$ field

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$ for $\vec{B}$ field

Electromagnetic Waves:

In general,
electromagnetic waves

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

where ψ represents $\vec{E}$ or $\vec{B}$
or any of their components

3 Solutions for Waves:

- ✓ A plane wave

  satisfies wave equation in Cartesian coordinates

- ✓ A spherical wave

  satisfies wave equation in spherical polar coordinates

- ✓ A cylindrical wave

  satisfies wave equation in cylindrical coordinates

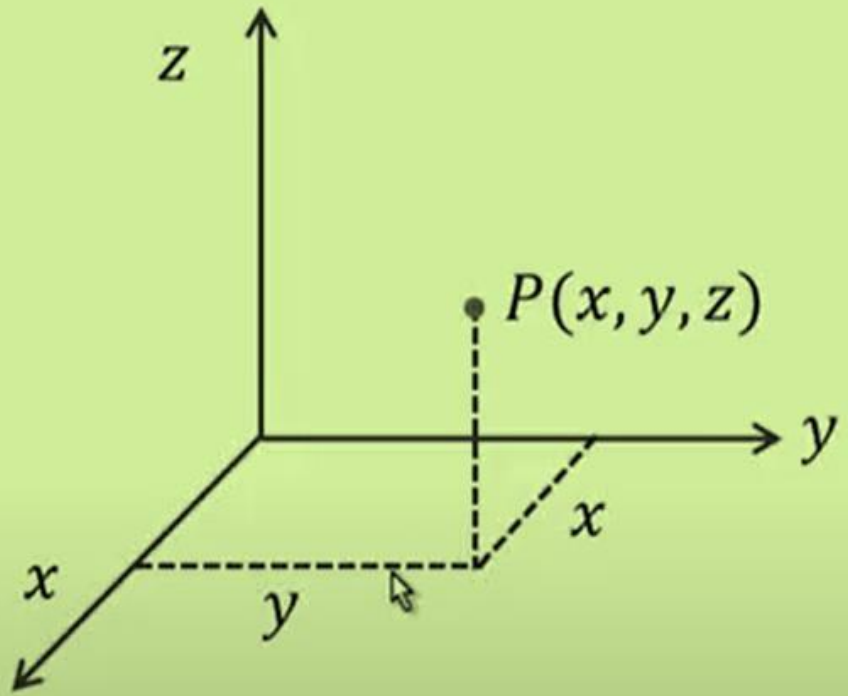
Solution of 3D Wave Equation:

- ✓ Plane Wave Solution

  wave equation in Cartesian coordinates

  waves from a source at large distance

Cartesian coordinates



Solution of 3D Wave Equation:

In Cartesian coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Separation of variables

$$\psi(x, y, z, t) = X(x)\,Y(y)\,Z(z)\,T(t)$$

Substituting for ψ we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{1}{C^2}\left(\frac{1}{T}\frac{\partial^2 \psi}{\partial t^2}\right)$$

- ✓ Variables are separated out
- ✓ Each variable-term independent
- ✓ And must be a constant

So we may write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2 \qquad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2$$

where we use

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = k^2$$

Solutions are then : $X(x) = e^{\pm i\, k_x\, x}$ and the time part :

$$Y(y) = e^{\pm i\, k_y\, y} \qquad T(t) = e^{\pm i\omega t}$$

$$Z(z) = e^{\pm i\, k_z\, z}$$

The total solution is :

$$\psi\,(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

$$= A e^{\,i\left[\left(k_x\, x + k_y\, y + k_z\, z\right) \pm \omega t\right]}$$

$$= A e^{i\left(\vec{k}.\vec{r} \pm \omega t\right)} \quad \Leftarrow \text{ plane wave}$$

A plane wave:

$$\psi(x,y,z,t) = Ae^{i(\vec{k}.\vec{r} \pm \omega t)}$$

$$= Ae^{i[(k_x x + k_y y + k_z z) \pm \omega t]}$$

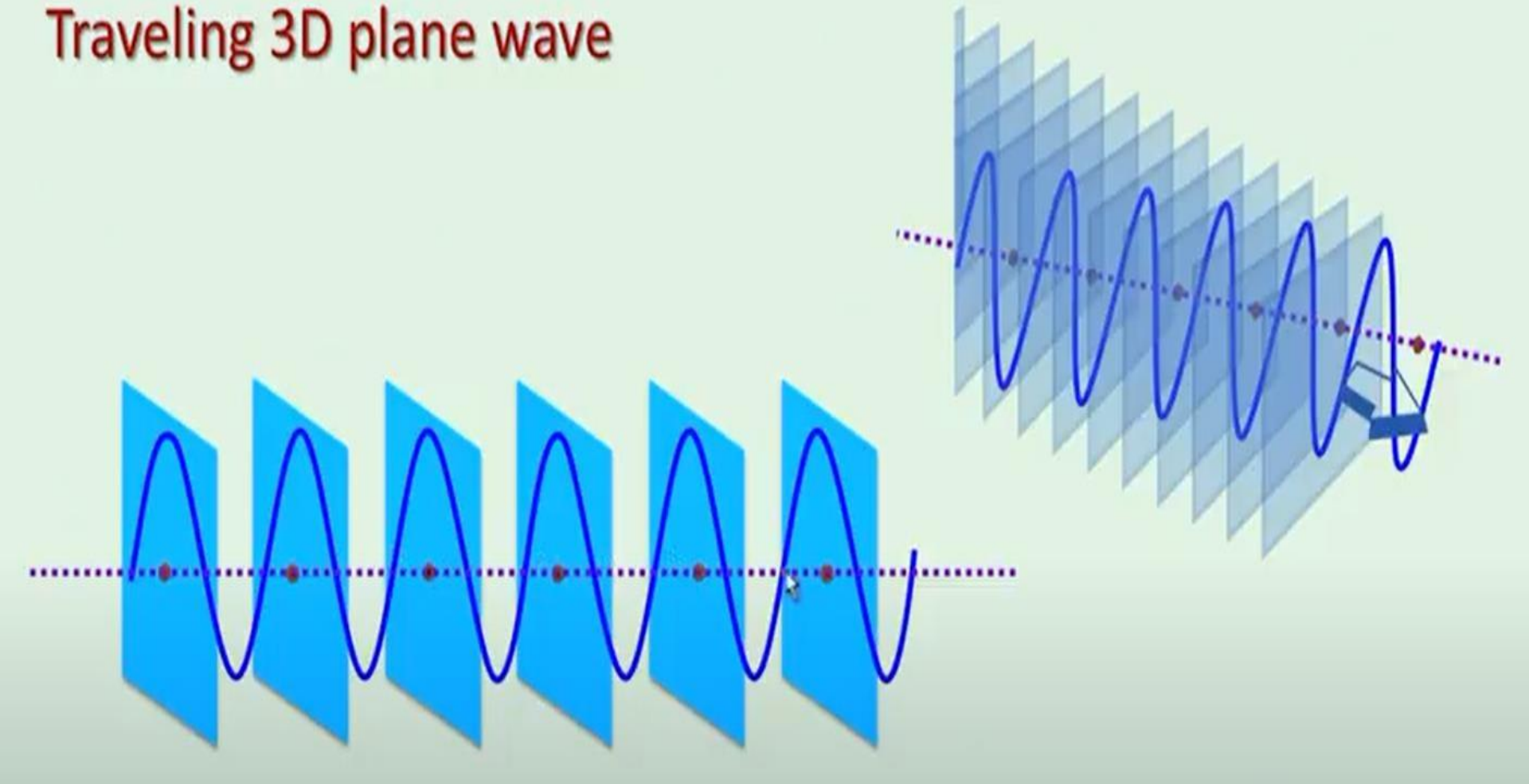
A wave whose phase is constant over a plane surface at any instant of time:

$$(k_x x + k_y y + k_z z) \pm \omega t = const$$

$$k_x x + k_y y + k_z z = const \quad \Longrightarrow \text{ equation of a plane surface}$$

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = const \qquad l = \frac{1}{k_x}; \text{ etc. are the direction cosines}$$

Traveling 3D plane wave



Traveling 3D plane wave $\psi(\vec{r},t) = Ae^{i(\vec{k}.\vec{r} \pm \omega t)}$

$$\vec{E} = \hat{j}E_x + \hat{\imath}E_y + \hat{k}E_z$$

$$\vec{B} = \hat{\imath}B_x + \hat{j}B_y + \hat{k}B_z$$

$$E_x = E_{x0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$E_y = E_{y0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$E_z = E_{z0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$B_x = B_{x0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$B_y = B_{y0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$B_z = B_{z0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

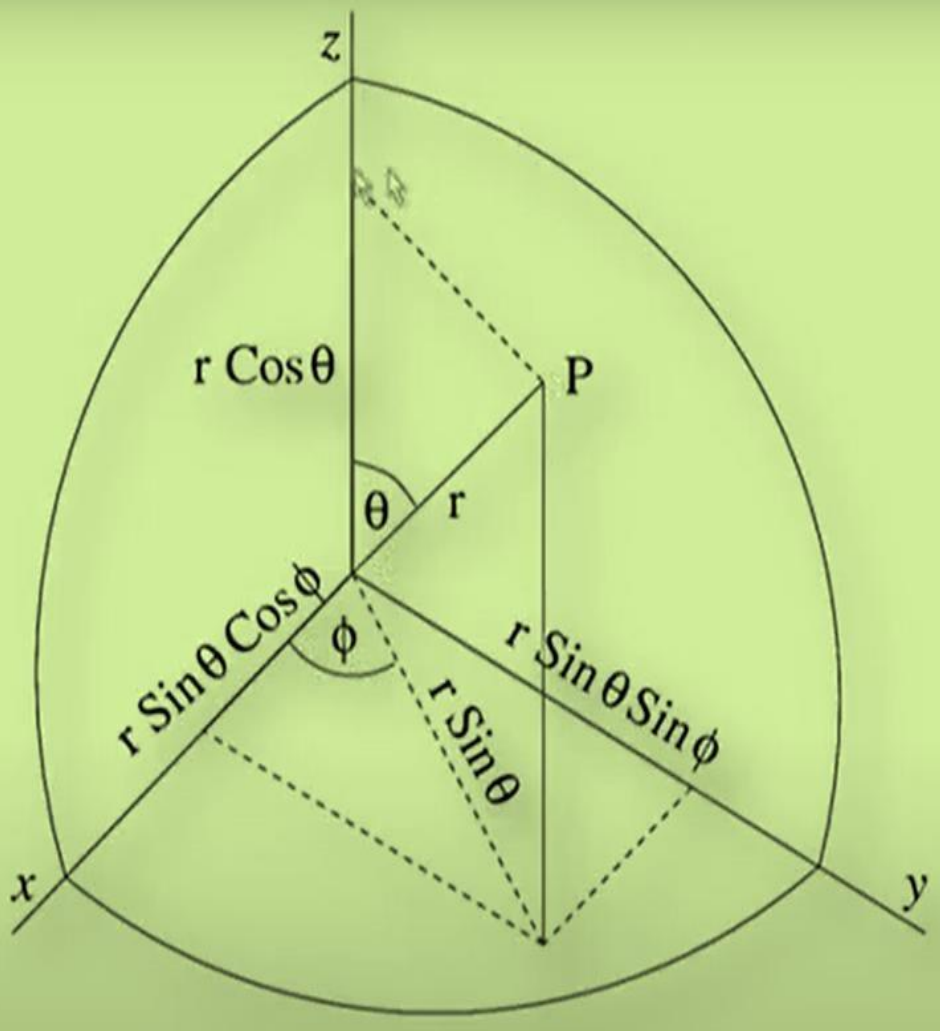
Solution of 3D Wave Equation:

- ✓ Spherical Wave Solution

  wave equation in Spherical Polar coordinates

  waves from a point source

spherical coordinates



Solution of 3D Wave Equation:

In Spherical coordinates

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\nabla^2 \equiv \underbrace{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}}_{\text{radial part}} + \underbrace{\frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}}_{\text{angular part}}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

assuming spherical symmetry

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \cancel{\frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta}} + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}} + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}}$$

Thus, the Laplacian reduces to the form

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2}$$

Therefore, the wave equation becomes

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2(r\psi)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2(r\psi)}{\partial t^2}$$

$$\frac{\partial^2(r\psi)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2(r\psi)}{\partial t^2}$$

choose a function: $r\psi = u$

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where $r\psi = u$ satisfies one dimensional wave equation

the 1-d wave equation: $\frac{\partial^2 u}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

yields plane wave solution for the transformed function $u = r\psi$

$$u = r\psi(r,t) = f(r - vt) + g(r + vt)$$

---

so the solution for ψ is

$$\psi(r,t) = \frac{1}{r} f(r - vt) + \frac{1}{r} g(r + vt)$$

$$\frac{\partial^2(r\psi)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2(r\psi)}{\partial t^2}$$

solution of above equation is thus

$$r\psi(r,t) = Ae^{i(kr \pm \omega t)} = Ae^{ik(r \pm vt)}$$

---

hence solution for ψ takes the form

$$\psi(r,t) = \frac{A}{r}e^{i(kr - \omega t)} \Rightarrow \text{spherical wave}$$

A spherical wave:

$$\psi(r,t) = \frac{A}{r} e^{i(kr - \omega t)}$$

$$= \frac{A}{r} e^{i(|\vec{k}||\vec{r}| - \omega t)}$$

A wave having same phase over a spherical surface at any instant of time:

$$k\sqrt{x^2 + y^2 + z^2} - \omega t = const$$

$$k\sqrt{x^2 + y^2 + z^2} = const \quad \text{at a given time } t$$

$$x^2 + y^2 + z^2 = const \quad \Longrightarrow \quad \text{equation of a sphere}$$

wave equation solution: spherical wave

Alternatively

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)$$

The wave equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Putting $\psi(r) = \dfrac{u(r)}{r}$

Then $\dfrac{\partial \psi}{\partial r} = \dfrac{1}{r}\dfrac{\partial u}{\partial r} - \dfrac{u}{r^2} \quad \Rightarrow \quad r^2 \dfrac{\partial \psi}{\partial r} = r\dfrac{\partial u}{\partial r} - u$

Hence $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \psi}{\partial r}\right) = \dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r} - u\right)$

$$= \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} - \frac{\partial u}{\partial r}$$



therefore $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right)=\frac{1}{r}\frac{\partial^2 u}{\partial r^2}$$

and wave equation transforms to

$$\frac{1}{r}\frac{\partial^2 u}{\partial r^2}=\frac{1}{v^2}\frac{1}{r}\frac{\partial^2 u}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial r^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Separation of variables: $u(r,t) = R(r)\,T(t)$

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \quad \Rightarrow \quad \frac{1}{R}\frac{\partial^2 R}{\partial r^2} = \frac{1}{v^2}\frac{1}{T}\frac{\partial^2 T}{\partial r^2} = -k^2$$

Solutions are $R(r) = e^{\pm ikr}$ $T(t) = e^{\pm i\omega t}$ $\omega = kv$

Total solution: $u(r) = e^{i(kr \pm \omega t)}$

Solution for ψ is :

$$\psi(r) = \frac{1}{r} e^{i(kr \pm \omega t)}$$ spherical wave

General solution :

$$\psi(r) = \frac{1}{r} e^{i(kr - \omega t)} + \frac{1}{r} e^{i(kr - \omega t)}$$

Outgoing waves Incoming wave

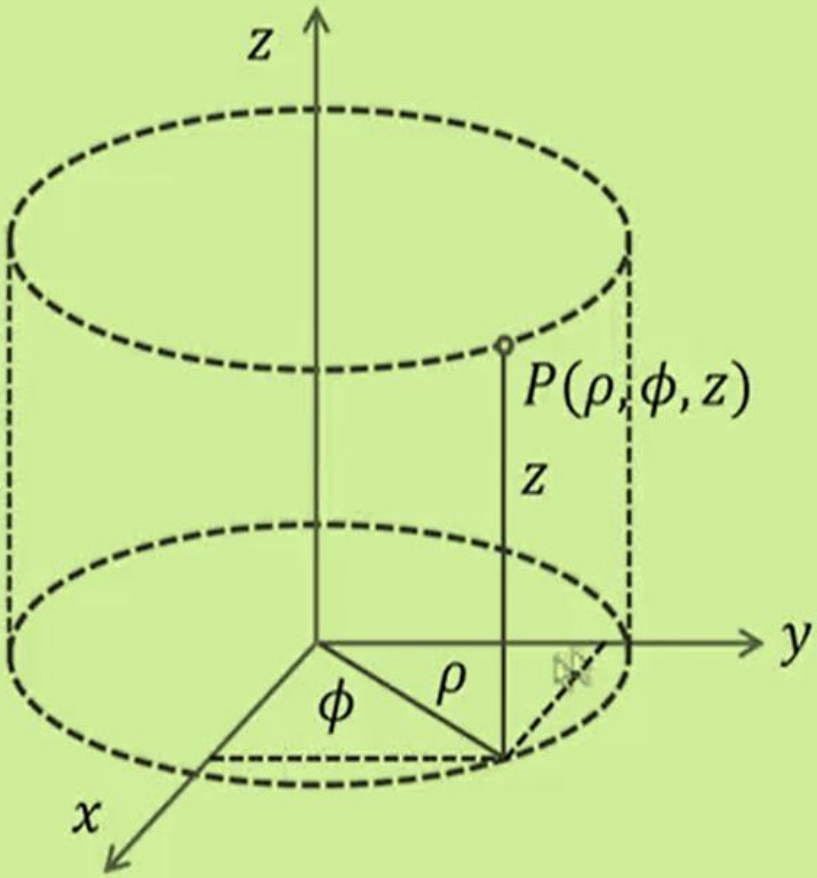
Solution of 3D Wave Equation:

- ✓ Cylindrical Wave Solution

  wave equation in Cylindrical coordinates

  waves from a line source

cylindrical coordinates



Solution of 3D Wave Equation:

In Cylindrical Coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

This Laplacian in wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

- ✓ with angular and azimuthal symmetry, the Laplacian simplifies

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)$$

- ✓ and the wave equation takes the form $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)=\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

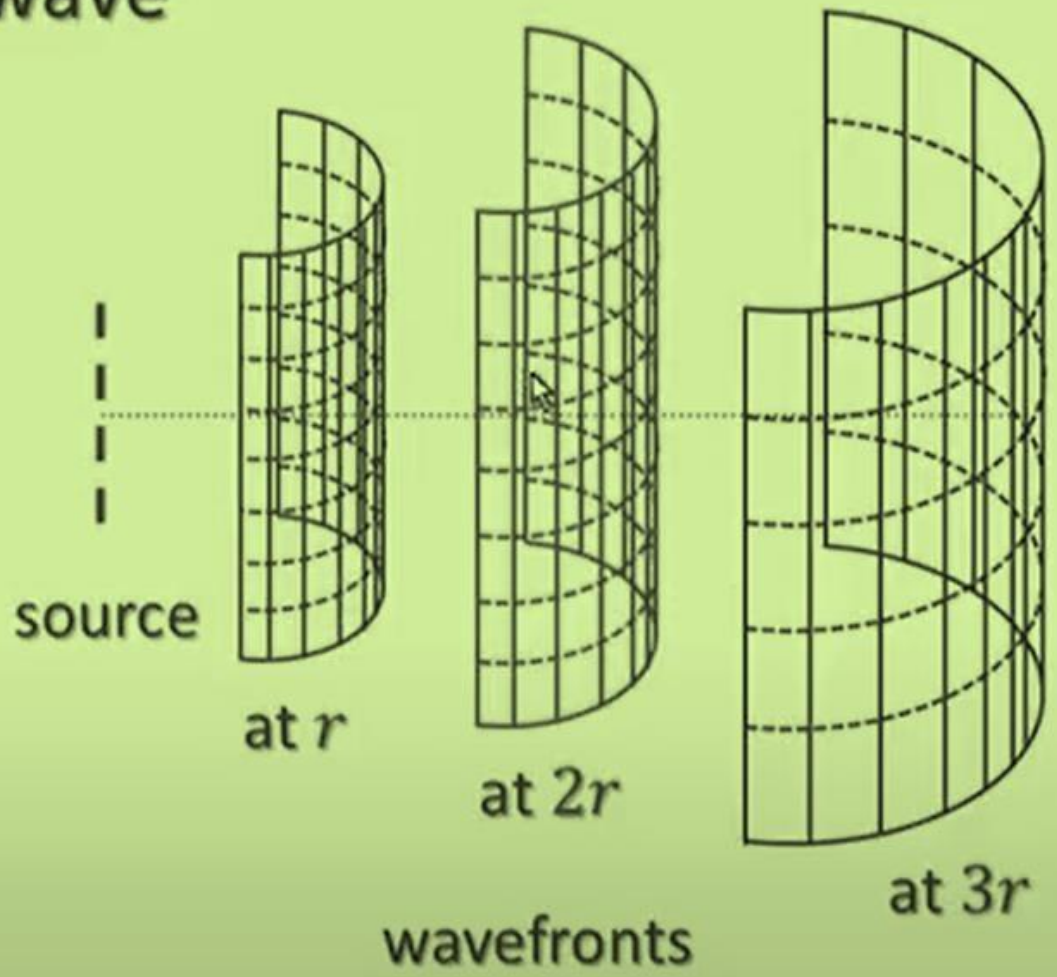
- ✓ The solutions are the Bessel functions

$$\psi(\rho,t)\rightarrow B(k\rho)e^{i\omega t}$$

- ✓ For large ρ, they are approximated as

$$\psi(\rho,t)\approx\frac{1}{\sqrt{\rho}}cos(k\rho-\omega t)$$

Traveling spherical wave



Contents:

- ✓ Plane EM waves in Free Space
- ✓ Relations and directions of $\vec{E}$, $\vec{B}$ and $\vec{k}$ vectors
- ✓ Polarisation of the EM waves

EM wave in vacuum

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

where ψ represents $\vec{E}$ or $\vec{B}$

or any of their components

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$$

plane wave solution

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

where ψ represents $\vec{E}$ or $\vec{B}$

or any of their components

$$\psi(x,y,z,t) = \psi_0 e^{i(\vec{k}.\vec{r} - \omega t)}$$

$$= \psi_0 e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

plane wave solution in vacuum

$$\psi(x, y, z, t) = \psi_0 e^{i(\vec{k}.\vec{r} - \omega t)}$$

$$= \psi_0 e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

where ψ represents $\vec{E}$ or $\vec{B}$ or any of their components

$\vec{E}$ and $\vec{B}$ fields and their components

$$\vec{E} = \hat{\imath}E_x + \hat{\jmath}E_y + \hat{k}E_z \qquad \vec{B} = \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z$$

$$E_x = E_{x0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]} \qquad B_x = B_{x0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$E_y = E_{y0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]} \qquad B_y = B_{y0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$E_z = E_{z0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]} \qquad B_z = B_{z0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

Maxwell's Equation: $\nabla \cdot \vec{E} = 0$

$$\nabla \cdot \vec{E} = \left(\hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\imath} E_x + \hat{\jmath} E_y + \hat{k} E_z \right)$$

$$= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)$$

the $\vec{E}$ field components

$$E_x = E_{x0}\, e^{\,i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$\frac{\partial E_x}{\partial x} = ik_x\, E_{x0}\, e^{\,i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$= ik_x E_x$$

so for all 3 components

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \left(ik_x E_x + ik_y E_y + ik_z E_z\right) = i\vec{k} \cdot \vec{E}$$

Maxwell's Equation: $\nabla \cdot \vec{E} = 0$

$$\nabla \cdot \vec{E} = \left(\hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\imath} E_x + \hat{\jmath} E_y + \hat{k} E_z \right)$$

$$= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)$$

$$= \left(ik_x E_x + ik_y E_y + ik_z E_z \right) = i\vec{k} \cdot \vec{E}$$

So we obtain that: $\vec{k} \cdot \vec{E} = 0$

$$\nabla \cdot \vec{E}(\vec{r},t) = \vec{k} \cdot \vec{E} = 0$$

wave vector $\vec{k}$ is perpendicular to $\vec{E}$

In the same way .....

$$\nabla \cdot \vec{B}(\vec{r},t) = \vec{k} \cdot \vec{B} = 0$$

wave vector $\vec{k}$ is perpendicular to $\vec{B}$

Maxwell's Equation: $\nabla \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

LHS :

$$\nabla \times \vec{E} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{pmatrix}$$

$$\frac{\partial E_z}{\partial y} = ik_y E_{z0} e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$= ik_y E_z$$

$$(\nabla \times \vec{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = ik_y E_z - ik_z E_y = i(\vec{k} \times \vec{E})_x$$

taking all 3 components $\nabla \times \vec{E} = i(\vec{k} \times \vec{E})$

Maxwell's Equation: $\nabla \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

RHS :

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z\right)$$

$$= -i\omega\left(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z\right)$$

$$= -i\omega\vec{B}$$

$$\frac{\partial B_x}{\partial t} = i\omega\, B_{x0}\, e^{i[(k_x x + k_y y + k_z z) - \omega t]}$$

$$= i\omega\, B_x$$

therefore, this term $\dfrac{\partial \vec{B}}{\partial t} = -i\omega\vec{B}$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$$

$$\nabla \times \vec{E} = i(\vec{k} \times \vec{E}) \qquad \frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B} \Rightarrow \vec{k} \times \vec{E} = \omega \vec{B}$$

$\vec{B}$ is perpendicular to $\vec{E}$

$\vec{B}$ is perpendicular to $\vec{k}$

$$\hat{k} \times \vec{E} = \frac{\omega}{k}\vec{B} = c\vec{B}$$

In the same way .....

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \varepsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$$

$$\vec{B} \times \vec{k} = \frac{\omega}{c^2}\vec{E}$$

$$\nabla \times \vec{B} = i(\vec{k} \times \vec{B}) \qquad \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = -\frac{i\omega}{c^2}\vec{E} \Rightarrow \qquad \vec{k} \times \vec{B} = -\frac{\omega}{c^2}\vec{E}$$

$\vec{B}$ is perpendicular to $\vec{E}$

$\vec{k}$ is perpendicular to $\vec{E}$

$$\vec{B} \times \hat{k} = \frac{\omega}{kc^2}\vec{E} = \frac{1}{c}\vec{E}$$

Orientation of the fields

$\vec{B}$ is perpendicular to $\vec{E}$

$\vec{B}$ is perpendicular to $\vec{k}$

$\vec{k}$ is perpendicular to $\vec{E}$

$\vec{B}$, $\vec{k}$ and $\vec{E}$ form a right handed Cartesian co-ordinate system

so we see .....

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

$$\vec{B} \times \vec{k} = \frac{\omega}{c^2} \vec{E}$$

take $\vec{k}$ along z and $\vec{E}$ along x

so $\vec{B}$ along y and $\vec{k}$ along z

$$k_z \times E_x = \omega B_y$$

$$B_y \times k_z = \frac{\omega}{c^2} E_x$$

then $\vec{B}$ happens to be along y

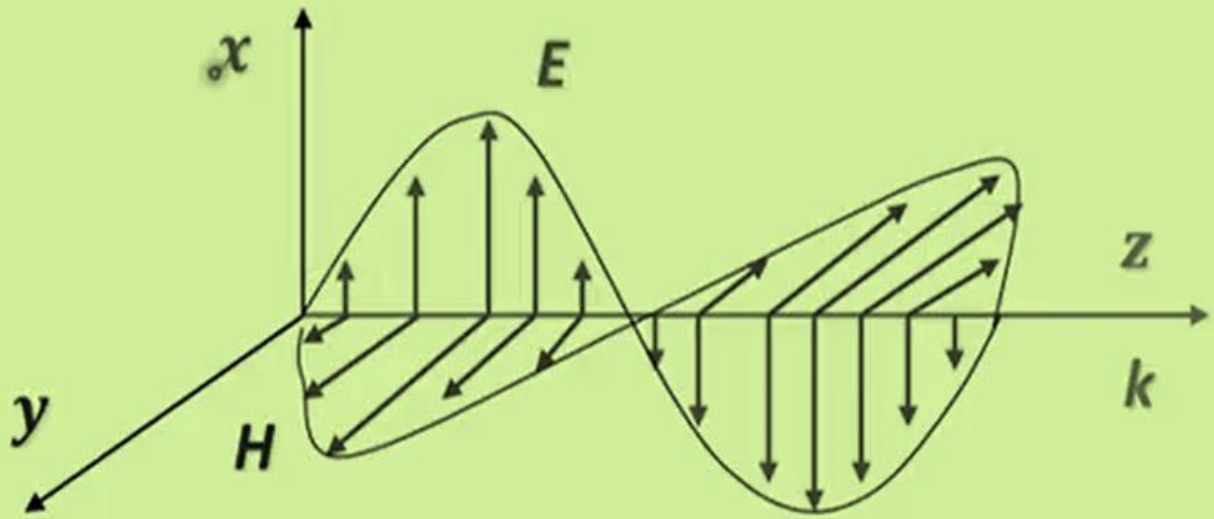
then $\vec{E}$ happens to be along x

Consistent !

$$\{\vec{E}, \vec{B}, \vec{k}\} \rightarrow \{x, y, z\}$$

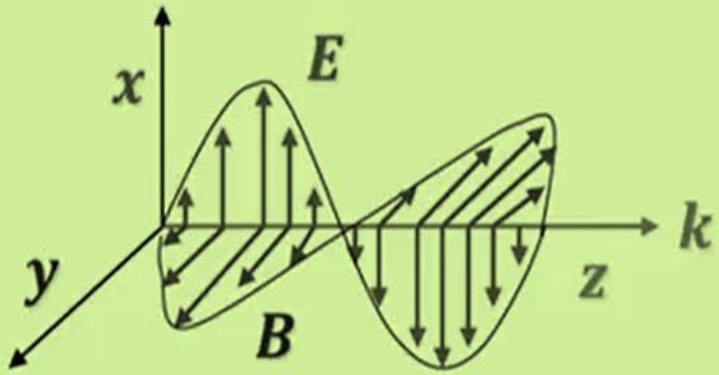
right handed triad

For most dielectrics: $\tilde{\mu} \approx \mu_0$ and $\vec{B} = \mu_0 \vec{H}$



$\vec{E}, \vec{H}, \vec{k}$ form a right handed Cartesian co-ordinate system

Consider:



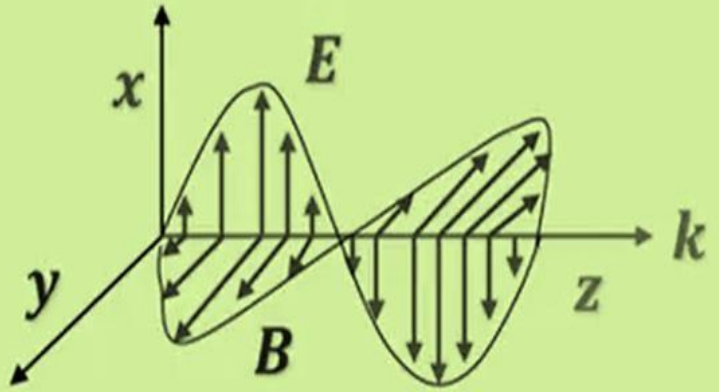
propagation along $z \rightarrow \vec{k} = \hat{z}k$

assume $\vec{E}$ along $x \rightarrow \vec{E} = \hat{x}E_0 \sin(\omega t - kz)$

then the $\vec{B}$ along $y \rightarrow \vec{B} = \hat{y}B_0 \sin(\omega t - kz)$

So $\vec{E}$ and $\vec{B}$ are in phase ⇨ linearly polarised wave

Consider:



propagation along $z \rightarrow \vec{k} = \hat{z}k$

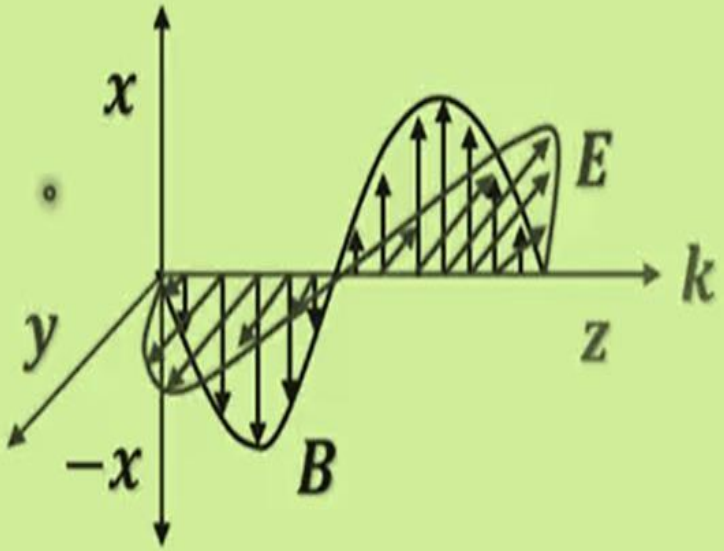
assume $\vec{E}$ along $x \rightarrow \vec{E} = \hat{x}E_0 \sin(\omega t - kz)$

then the $\vec{B}$ along $y \rightarrow \vec{B} = \hat{y}B_0 \sin(\omega t - kz)$

So $\vec{E}$ and $\vec{B}$ are in phase ⇨ linearly polarised wave

Next consider:

another wave: y −polarised



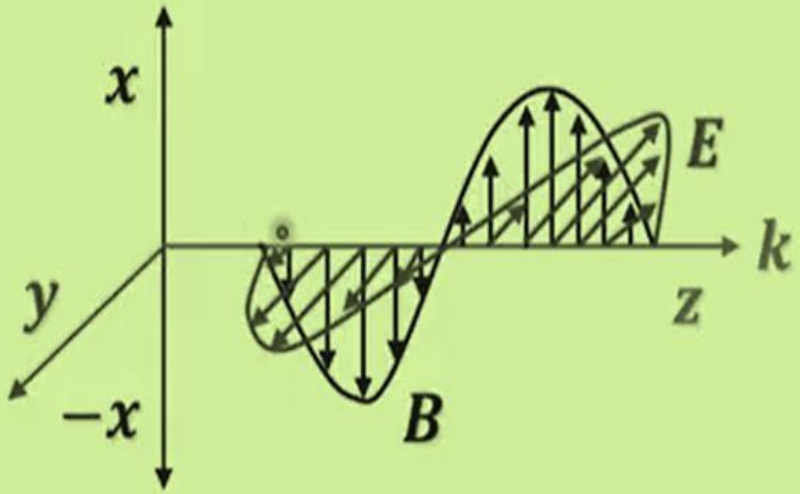
$$\vec{E} = \hat{y}E_0 \sin(\omega t - kz)$$

$$\vec{B} = -\hat{x}B_0 \sin(\omega t - kz)$$

Next consider:

another wave: y −polarised

with an additional phase of $\frac{\pi}{2}$



$$\vec{E} = \hat{y}E_0 \sin\left(\omega t - kz + \frac{\pi}{2}\right) \quad = \hat{y}E_0 \cos(\omega t - kz)$$

$$\vec{B} = -\hat{x}B_0 \sin\left(\omega t - kz + \frac{\pi}{2}\right) \quad = -\hat{x}B_0 \cos(\omega t - kz)$$

Maxwell's equations are linear

Linear superposition of these $x-$ and $y-$polarised waves

$$\vec{E} = \hat{x}E_0 \sin(\omega t - kz)$$

$$\vec{B} = \hat{y}B_0 \sin(\omega t - kz)$$

$x-$polarised

$$\vec{E} = \hat{y}E_0 \cos(\omega t - kz)$$

$$\vec{B} = -\hat{x}B_0 \cos(\omega t - kz)$$

$y-$polarised

Maxwell's equations are linear

Linear superposition of these $x-$ and $y-$polarised waves

$$\vec{E} = \hat{x}E_0 \sin(\omega t - kz) + \hat{y}E_0 \cos(\omega t - kz)$$

$$\vec{B} = \hat{y}B_0 \sin(\omega t - kz) - \hat{x}B_0 \cos(\omega t - kz)$$

Consider a plane perpendicular to $\vec{k}$ at $z = 0$

All points on the plane time, dependence of $\vec{E}$, $\vec{B}$:

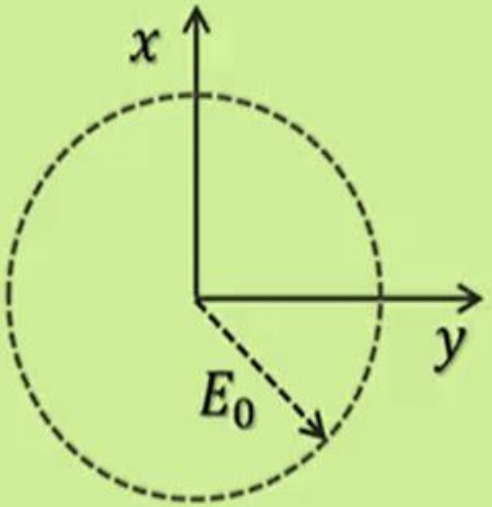
$$E_x = E_0 \sin \omega t \quad \text{and} \quad E_y = E_0 \cos \omega t$$

$$B_x = -B_0 \cos \omega t \quad \text{and} \quad B_y = E_0 \sin \omega t$$

so the resultant wave

$$E^2 = E_x{}^2 + E_y{}^2 = E_0{}^2$$

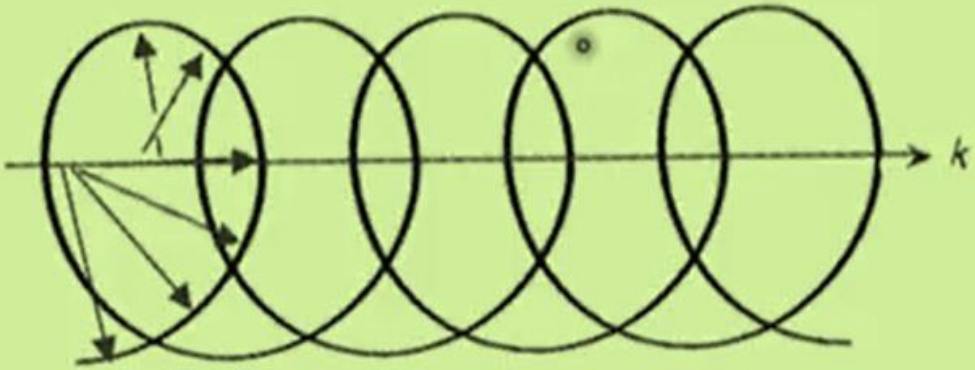
$$B^2 = B_x{}^2 + B_y{}^2 = B_0{}^2$$

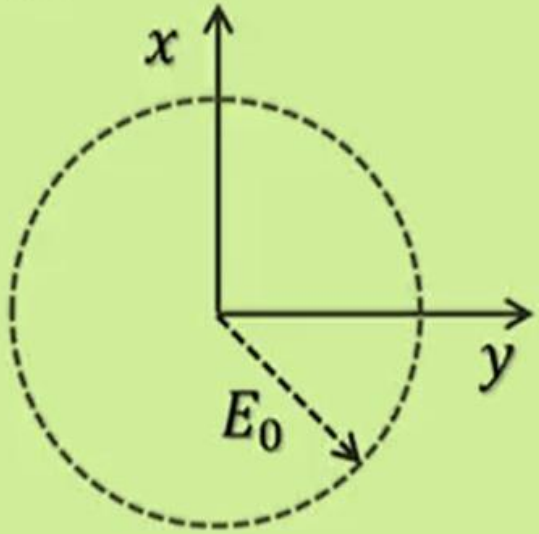


the tip of $\vec{E}, \vec{B}$ vectors rotate on the circumference of a circle

circularly polarised wave

the wave is travelling along z direction





along z the tip of each $\vec{E}, \vec{B}$ vector traces an advancing helix

circularly polarised wave

at $z = 0$

$E_x = E_0 \sin \omega t$ and $E_y = E_0 \cos \omega t$

$B_x = -B_0 \cos \omega t$ and $B_y = E_0 \sin \omega t$

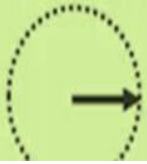
at $t = 0$ $E_x = 0$ $E_y = E_0$



at $t = \frac{\pi}{4}\frac{1}{\omega}$ $E_x = \frac{1}{\sqrt{2}}E_0$ $E_y = \frac{1}{\sqrt{2}}E_0$



at $t = 2\frac{\pi}{4}\frac{1}{\omega}$ $E_x = E_0$ $E_y = 0$



right-circularly polarized

- ✓ looking at the source
  $\vec{E}$ of the wave coming towards you is seen rotating anticlockwise
  the wave is right-circularly polarized

- ✓ looking at the source
  $\vec{E}$ is seen rotating clockwise
  the wave is left-circularly polarized

For a combination of superposition of $x-$ and $y-$polarised waves as

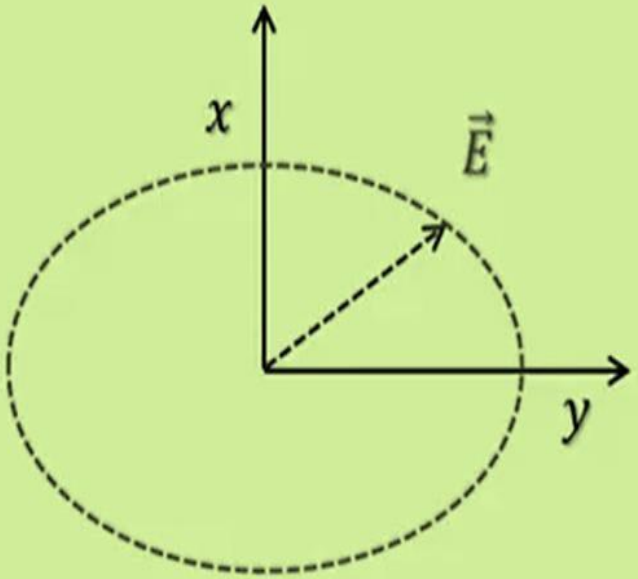
$$\vec{E} = \hat{x}E_0 \cos(\omega t - kz) - \hat{y}E_0 \sin(\omega t - kz)$$

$$\vec{B} = \hat{y}B_0 \cos(\omega t - kz) + \hat{x}B_0 \sin(\omega t - kz)$$

left-circularly polarized

when the amplitudes of E_x and E_y are not the same

in the resultant wave, the tip of $\vec{E}$ describes an ellipse



elliptically polarised wave

Contents

- ✓ Energy transport in EM waves
- ✓ Poynting Vector and **Poynting's** Theorem
- ✓ Energy transport in dielectric and conductor

Poynting's Theorem

- ✓ Poynting's theorem concerns the conservation of energy for a given volume in space.
- ✓ Poynting's theorem is a consequence of Maxwell's equations.

Power flow associated with EM wave

Let's begin with
Maxwell's Curl Equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Recall the vector identity

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = \vec{H}.(\vec{\nabla}\times\vec{E}) - \vec{E}.(\vec{\nabla}\times\vec{H})$$

$$\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \end{cases}$$

Plug in the curl equations

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{J}.\vec{E} - \vec{E}.\frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}.(\vec{E} \times \vec{H}) = -\vec{j}.\vec{E} - \vec{E}.\frac{\partial \vec{D}}{\partial t} - \vec{H}.\frac{\partial \vec{B}}{\partial t}$$

Integrating over a volume V bounded by a closed surface S

$$\int_V \vec{\nabla}.(\vec{E} \times \vec{H})\, dV = -\int_V \vec{j}.\vec{E}\, dV - \int_V \left(\vec{E}.\frac{\partial \vec{D}}{\partial t} + \vec{H}.\frac{\partial \vec{B}}{\partial t}\right) dV$$

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = -\vec{j}.\vec{E} - \vec{E}.\frac{\partial\vec{D}}{\partial t} - \vec{H}.\frac{\partial\vec{B}}{\partial t}$$

Integrating over a volume V bounded by a closed surface S

$$\int_V \vec{\nabla}.(\vec{E}\times\vec{H})\, dV = -\int_V \vec{j}.\vec{E}\, dV - \int_V \left(\vec{E}.\frac{\partial\vec{D}}{\partial t} + \vec{H}.\frac{\partial\vec{B}}{\partial t}\right) dV$$

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = -\vec{j}.\vec{E} - \vec{E}.\frac{\partial\vec{D}}{\partial t} - \vec{H}.\frac{\partial\vec{B}}{\partial t}$$

Integrating over a volume V bounded by a closed surface S

$$\int_V \vec{\nabla}.(\vec{E}\times\vec{H})\,dV = -\int_V \vec{j}.\vec{E}\,dV - \int_V \left(\vec{E}.\frac{\partial\vec{D}}{\partial t} + \vec{H}.\frac{\partial\vec{B}}{\partial t}\right) dV$$

Energy transport: Poynting Vector

Now use Gauss's Theorem : $\int_V \vec{\nabla}.\vec{F}\, dV = \int_S \vec{F}.\hat{n}\, dA$

$$\int_V \vec{\nabla}.(\vec{E}\times\vec{H})\, dV = -\int_V \vec{j}.\vec{E}\, dV - \int_V \left(\vec{E}.\frac{\partial\vec{D}}{\partial t} + \vec{H}.\frac{\partial\vec{B}}{\partial t}\right) dV$$

$$\int_S (\vec{E}\times\vec{H}).\hat{n}\, dA = -\int_V \vec{j}.\vec{E}\, dV - \int_V \left(\vec{E}.\frac{\partial\vec{D}}{\partial t} + \vec{H}.\frac{\partial\vec{B}}{\partial t}\right) dV$$

Define a vector $\vec{S}$: The Poynting Vector

$\vec{S} \equiv \vec{E} \times \vec{H}$ $\vec{S}$ is perpendicular to $\vec{H}$ and $\vec{E}$

$$\int_S \vec{S}.\hat{n}\, dA = -\int_V \vec{J}.\vec{E}\, dV - \int_V \left(\vec{E}.\frac{\partial \vec{D}}{\partial t} + \vec{H}.\frac{\partial \vec{B}}{\partial t}\right) dV$$

- ✓ Poynting Vector measures the rate of energy transport
- ✓ In a direction perpendicular to $\vec{H}$ and $\vec{E}$: along propagation

Recall that

Using the constitutive relations

Energy density associated with

Electric Field : $w_E = \frac{1}{2}(\vec{E}.\vec{D}) = \frac{1}{2}\varepsilon.E^2$

Magnetic Field : $w_B = \frac{1}{2}(\vec{H}.\vec{B}) = \frac{1}{2}\mu.H^2$

Terms in Poynting's Theorem

energy density associated
with electric and magnetic fields

$$w_E = \frac{1}{2}(\vec{E}.\vec{D})$$

$$w_B = \frac{1}{2}(\vec{H}.\vec{B})$$

$$\vec{E}.\frac{\partial \vec{D}}{\partial t} = \frac{1}{2}\varepsilon.\frac{\partial E^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}.\vec{D}) = \frac{\partial}{\partial t}(w_E)$$

$$\vec{H}.\frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\mu.\frac{\partial H^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}.\vec{B}) = \frac{\partial}{\partial t}(w_B)$$

Energy transport: Poynting Vector

Rewritten in the form:

$$\int_A \vec{S}.\hat{n}\ dA = -\int_V \vec{J}.\vec{E}\ dV - \int_V \frac{\partial}{\partial t}(w_B + w_E)dV$$

$$= -\int_V \vec{J}.\vec{E}\ dV - \frac{\partial}{\partial t}\int_V (w_B + w_E)dV$$

Terms in Poynting's Theorem

The term: $\int_V (w_B + w_E) dV$

$$= \int_V \left(\frac{1}{2} \varepsilon . E^2 + \frac{1}{2} \mu . H^2 \right) dV$$

represents the *total electromagnetic energy stored* in volume *V*

Terms in Poynting's Theorem

The term : $$\int_S \vec{S}.\hat{n}\ dA = \int_S (\vec{E} \times \vec{H}).\hat{n}\ dA$$

represents the *flow of instantaneous power* out of the volume *V* through the surface *S*

Energy transport: Poynting Vector

Rewritten in the form:

$$\oint_A \vec{S}.\hat{n}\ dA = -\int_V \vec{J}.\vec{E}\ dV - \int_V \frac{\partial}{\partial t}(w_B + w_E)dV$$

Total power flowing out of a volume V

Dissipated power in volume V

Total electromagnetic energy change in volume V

$$or, \qquad \vec{\nabla}.\vec{S} + \frac{\partial}{\partial t}(w_B + w_E) = -\vec{J}.\vec{E}$$

Stokes theorem

Statement: The line integral of a vector ($\vec{A}$) taken around a closed curve (C) is equal to the surface integral of the curl of the same vector ($\vec{A}$) taken over surface(S) bounded by same closed path (C).

$$\oint_l \vec{A}\, dr = \iint_S curl\, \vec{A}\; ds$$

Maxwell's Equations

There are four fundamental equations of electromagnetism known as Maxwell's equations which represented in differential form as,

1) $\vec{\nabla} \cdot \vec{D} = \rho$

It represents Gauss law in electrostatics. Where $\vec{D}$ is electric displacement and ρ is the free charge density.

2) $\vec{\nabla} \cdot \vec{B} = 0$

It represents Gauss law in magnetostatics. Where $\vec{B}$ is magnetic induction.

3) $\vec{\nabla}\, X\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

It represents Faraday's law of electromagnetic induction. Where $\vec{E}$ is electric field intensity and $\vec{B}$ is magnetic induction.

4) $\vec{\nabla}\, X\, \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

It represents Maxwell's modification in Amperes's law. Where $\vec{E}$ is electric field intensity and $\vec{B}$ is magnetic induction.

Derivations of Maxwell's Equations

1) $div\, \vec{D} = \vec{\nabla} \cdot \vec{D} = \rho$

Consider a surface S bounding a volume V in dielectric medium under electric field. So we have two types of charges free charges and induced charges.

Let ρ and ρ' be the charge densities of free charges and induced charges.

According to Gauss law in electrostatics

Electric flux linked with closed surface,

$\phi_E = \frac{1}{\varepsilon_0} q$ (where q is net charge present on surface S)

$$\because\ \phi_E = \int_s \vec{E}\; ds\ \&\ q = \int_v (\rho + \rho')dv$$

$\therefore$ Above equation becomes

$$\int_s \vec{E}\; ds = \frac{1}{\varepsilon_0}\int_v (\rho + \rho')dv$$

$$\varepsilon_0 \int_s \vec{E}\; ds = \int_v \rho\; dv + \int_v \rho' dv$$

$$\because \rho' = -div\, \vec{P}$$

$\therefore$ Above equation becomes

$$\int_s \varepsilon_0 \vec{E}\; ds = \int_v \rho\; dv - \int_v div\, \vec{P}\; dv$$

Using Gauss Divergence theorem on left side to change surface integral to volume integral, we get

$$\int_v div\, (\varepsilon_0 \vec{E})\; dv = \int_v \rho\; dv - \int_v div\, \vec{P}\; dv$$

$$\int_v div\,(\varepsilon_0\vec{E}+\vec{P})\,dv = \int_v \rho\,dv$$

$\because \varepsilon_0\vec{E}+\vec{P}=\vec{D}$, Electric Displacement

$\therefore$ Above equation becomes

$$\int_v div\,\vec{D}\,dv = \int_v \rho\,dv$$

$$\int_v (div\,\vec{D}-\rho)\,dv = 0$$

Since this equation is true for all volume, this integral vanishes

$$\therefore div\,\vec{D}-\rho=0$$

$$div\,\vec{D}=\rho \quad \text{or } \vec{\nabla}\cdot\vec{D}=\rho$$

2) $\vec{\nabla}\cdot\vec{B}=0$

Experimental results shows that the number of magnetic field lines entering to any arbitrary closed surface is equal to number of magnetic field lines leaving from it. Therefore the flux of magnetic induction $\vec{B}$ across any closed surface is always zero. ie.

$$\int_s \vec{B}\,ds = 0$$

Using Gauss Divergence theorem on left side to change surface integral to volume integral, we get

$$\int_v div\,\vec{B}\,dv = 0$$

Since this equation is true for all volume, this integral vanishes

$$div\,\vec{B}=0 \text{ or } \vec{\nabla}\cdot\vec{B}=0$$

3) $\vec{\nabla}\,X\,\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

Faraday's law states that induced emf in a circuit is proportional to negative rate of change of magnetic flux linked with it. According the Faraday's law,

Induced emf, $e=-\frac{d\phi_E}{dt}$

But, magnetic flux, $\phi_E=\int_s \vec{B}\,ds$

Where S is surface of closed loop C

$$\therefore e=-\frac{d}{dt}\left(\int_s \vec{B}\,ds\right)$$

$$=-\left(\int_s \frac{\partial\vec{B}}{\partial t}\,ds \qquad \ldots\ldots\ldots(1)\right.$$

We know emf is work done in carrying a unit charge around a closed loop in an electric field. Therefore,

$$e=\int_c \vec{E}\,dl \qquad \ldots\ldots\ldots(2)$$

Comparing equation (1) & (2), we get

$$\int_c \vec{E}\, dl = -\int_s \frac{\partial \vec{B}}{\partial t}\, ds$$

Using Stokes theorem to change line integral into surface integral, we get,

$$\int_s curl\, \vec{E}\, ds = -(\int_s \frac{\partial \vec{B}}{\partial t}\, ds$$

$$\int_s \left(curl\, \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) ds = 0$$

Since this equation is true for all surface, this integral vanishes

$$\therefore curl\, \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$curl\, \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad or \quad \vec{\nabla} X \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

4) $\vec{\nabla} X \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Ampere's Circuital law states that the line integral of magnetic field $\vec{H}$ around any closed path or circuit is equal to the current enclosed by the path.

According to Ampere's law, $\int_c \vec{H}\, dl = I$

$\because I = \int_s \vec{J}\, ds$, Where J is current density

The above equation becomes

$$\int_c \vec{H}\, dl = \int_s \vec{J}\, ds$$

Where S is surface bounded by closed path C.

Using Stokes theorem to change line integral into surface integral, we get,

$$\int_s curl\, \vec{H}\, ds = \int_s \vec{J}\, ds$$

$$\int_s (curl\, \vec{H} - \vec{J}) ds = 0$$

Since this equation is true for all surface, this integral vanishes

$$\therefore curl\, \vec{H} - \vec{J} = 0$$

$curl\, \vec{H} = \vec{J}$ ..... (1)

But Maxwell found it to be incomplete for time varying electric field and assumed that a quantity called displacement current and corresponding displacement current density must be included.

ie. $\overrightarrow{J_D} = \frac{\partial \vec{D}}{\partial t}$

Therefore in equation (1), $\vec{J}$ must be replaced by $\vec{J} + \overrightarrow{J_D}$, therefore equation (1) becomes,

$curl\, \vec{H} = \vec{J} + \overrightarrow{J_D}$

$$curl\, \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Integral form of Maxwell's equations and their physical significance

1) $\int_s \vec{D}\, ds = q$

This Maxwell's equation signifies

i) The total flux of the electric displacement linked with the closed surface is equal to the total charge enclosed by the close surface.

ii) The positive charges becomes the source of flux of the electric displacement and the negative charges becomes the sink to flux of the electric displacement. Thereby this equation confirms the existence of positive and negative charges.

2) $\int_s \vec{B}\, ds = 0$

This Maxwell's equation signifies

i) The total outward flux of magnetic induction linked with a closed surface is zero. That is magnetic flux lines are continuous forming closed loops implying that there are no sink or sources in the case of magnetic flux.

ii) It confirms the non existence of magnetic monopole and hence establishes that the magnetic poles always appear as dipoles.

3) $\int_c \vec{E}\, dl = -\int_s \frac{\partial \vec{B}}{\partial t}\, ds$

This Maxwell's equation signifies;

Line integral of electric intensity around closed path ie. electromotive force is equal to the negative rate of change of magnetic flux linked with the path.

4) $\int_c \vec{H}\, dl = \int_s \left(J + \frac{\partial \vec{D}}{\partial t}\right) ds$

This Maxwell's equation signifies;

Line integral of magnetic field intensity around closed path ie. magnetomotive force is equal sum of the conduction current and the displacement current linked with that path.

Derivation of general plane wave equation in free space

In the free space, the charge density $\rho = 0$, conductivity $\sigma = 0$ and hence current density $J = \sigma E = 0$. Also $\varepsilon_r = \mu_r = 1$, therefore $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$

Hence Maxwell's equations for free space can be reduced to

$$\vec{\nabla} \cdot \vec{D} = \rho \;\; ; \;\; \vec{\nabla} \cdot \vec{E} = 0 \quad \ldots\ldots\ldots\ldots (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \;\; ; \;\; \vec{\nabla} \cdot \vec{H} = 0 \quad \ldots\ldots\ldots\ldots (2)$$

$$\vec{\nabla}\, X\, \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ldots\ldots\ldots\ldots (3)$$

$$\vec{\nabla}\, X\, \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla}\, X\, \vec{H} = 0 + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \ldots\ldots\ldots\ldots (4)$$

$$\vec{\nabla}\, X\, \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ldots\ldots\ldots\ldots (4)$$

From equation (3),

$$\vec{\nabla}\, X \left(\vec{\nabla}\, X\, \vec{E}\right) = \vec{\nabla}\, X \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\vec{\nabla}\, X \left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\therefore \quad = -\frac{1}{2}\int_v \left[\vec{E}\cdot(\vec{\nabla}\,X\,\vec{H}) - \vec{H}\cdot(\vec{\nabla}\,X\,\vec{E})\right] dv \ldots\ldots\ldots(2)$$

We know, $\vec{A}\,curl\,\vec{B} - \vec{B}\,curl\,\vec{A} = div\,(\vec{B}\,X\,\vec{A}) = -div\,(\vec{A}\,X\,\vec{B})$

Therefore eq^n (2) becomes

$$\frac{dW}{dt} = -\frac{1}{2}\int_v \left[-div\,(\vec{E}\,X\,\vec{H})\right] dv$$

$$= \frac{1}{2}\int_v \left[div\,(\vec{E}\,X\,\vec{H})\right] dv$$

Using Gauss Divergence theorem to change volume integral into surface integral, we get,

$$\frac{dW}{dt} = \int_s (\vec{E}\,X\,\vec{H})\,ds \quad \ldots\ldots\ldots\ldots(3)$$

This gives Poynting Theorem. In this vector $\vec{E}\,X\,\vec{H}$ is called as poynting vector and denoted by $\vec{P}$ and it along the direction of propagation.

$\therefore \vec{P} = \vec{E}\,X\,\vec{H} \ldots\ldots\ldots\ldots(4)$

Physical significance

1) $\vec{P}$ represents an amount of energy of electromagnetic wave per unit area per unit time which is along the direction of propagation of electromagnetic wave.
2) Though $\overrightarrow{E}$ & $\vec{H}$ being oscillatory, change in direction, $\vec{P}$ remains unidirectional.
3) The magnitude of $\vec{P}$ varies. It is maximum when $\overrightarrow{E}$ & $\vec{H}$ are maximum and zeor when $\overrightarrow{E}$ & $\vec{H}$ are zero.

Difference between differential and integral form of Maxwell Equation



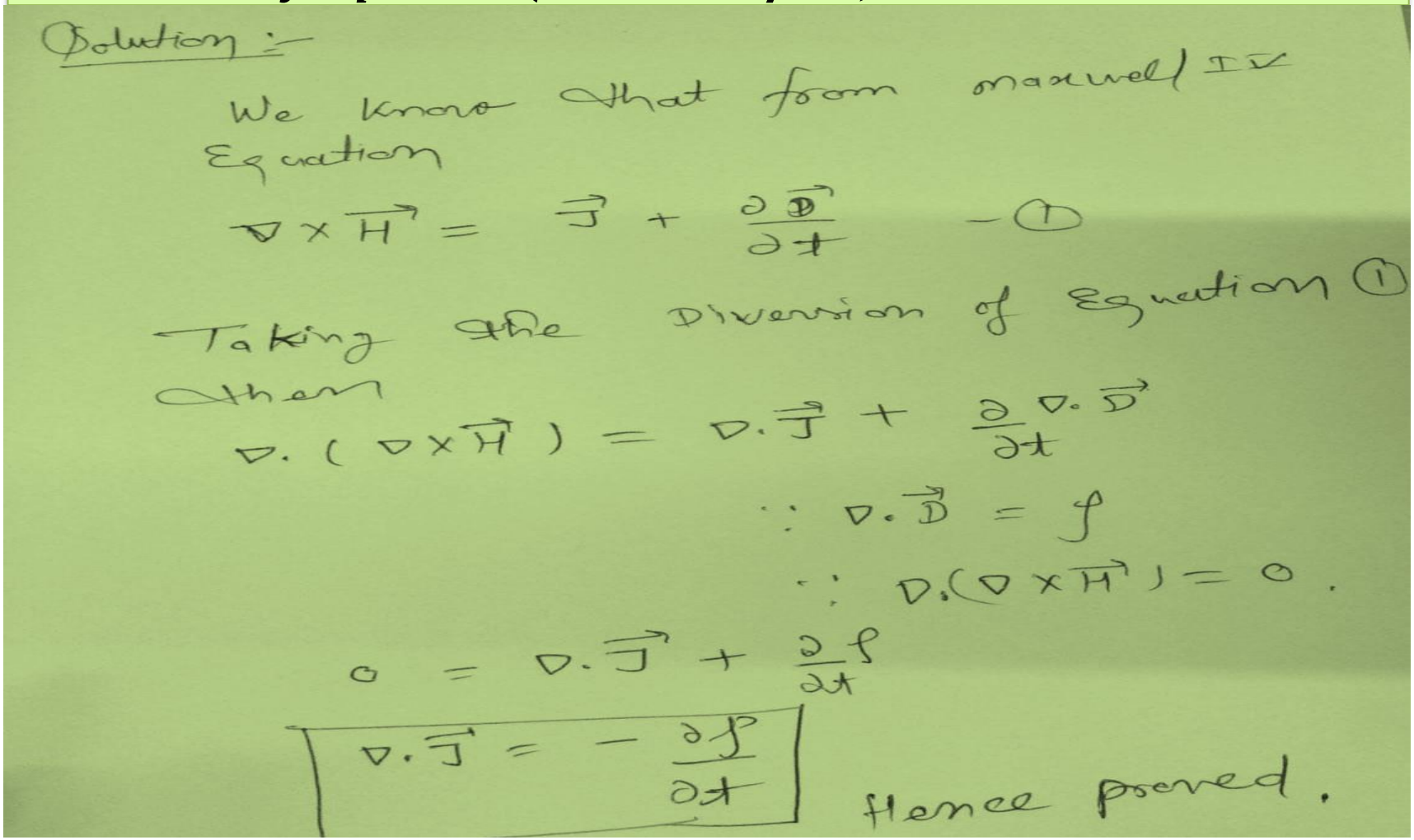
- The equations are entirely equivalent, as can be proven using Gauss' and Stokes' theorems.

- The integral forms are most useful when dealing with macroscopic problems with high degrees of symmetry (e.g. spherical or axial symmetry; or, following on from comments below, a line/surface integrals where the field is either parallel or perpendicular to the line/surface element).

- The differential forms are strictly local - they deal with charge and current *densities* and fields *at a point* in space and time. The differential forms are *far* easier to manipulate when dealing with electromagnetic waves; they make it *far* easier to show that Maxwell's equations can be written in a covariant form, compatible with special relativity, and *far* easier to put into a computer to do numerical electromagnetism calculations.

Tutorial

Q.1. Using the differential form of Maxwell's equations, derive the continuity equation ($\nabla . J\sim = -\partial\rho/\partial t$)



Q.2. The electric field of a traveling electromagnetic wave is given by

$$E(z,t) = 10\cos(\pi \times 10^7 t + \pi z/15 + \pi/6) \quad \text{(V/m)}.$$

Determine (a) the direction of wave propagation, (b) the wave frequency f, (c) its wavelength λ, and (d) its phase velocity u_p.

Solution:

(a) $-z$-direction because the signs of the coefficients of t and z are both positive.

(b) From the given expression,

$$\omega = \pi \times 10^7 \quad \text{(rad/s)}.$$

Hence,

$$f = \frac{\omega}{2\pi} = \frac{\pi \times 10^7}{2\pi} = 5 \times 10^6 \text{ Hz} = 5 \text{ MHz}.$$

(c) From the given expression,

$$\frac{2\pi}{\lambda} = \frac{\pi}{15}.$$

Hence $\lambda = 30$ m.

(d) $u_p = f\lambda = 5 \times 10^6 \times 30 = 1.5 \times 10^8$ m/s.

Q.3

Compute the intensity of the standing electromagnetic wave given by

$$E_y(x,t) = 2E_0 \cos kx \cos \omega t, \quad B_z(x,t) = 2B_0 \sin kx \sin \omega t$$

Solution:

The Poynting vector for the standing wave is

$$\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} = \frac{1}{\mu_0}(2E_0 \cos kx \cos \omega t\, \hat{\mathbf{j}}) \times (2B_0 \sin kx \sin \omega t\, \hat{\mathbf{k}})$$

$$= \frac{4E_0B_0}{\mu_0}(\sin kx \cos kx \sin \omega t \cos \omega t)\hat{\mathbf{i}}$$

$$= \frac{E_0B_0}{\mu_0}(\sin 2kx \sin 2\omega t)\hat{\mathbf{i}}$$

The time average of S is

$$\langle S \rangle = \frac{E_0B_0}{\mu_0} \sin 2kx \langle \sin 2\omega t \rangle = 0$$

The result is to be expected since the standing wave does not propagate. Alternatively, we may say that the energy carried by the two waves traveling in the opposite directions to form the standing wave exactly cancel each other, with no net energy transfer.

Q.4. If ε and μ assume negative values in a certain frequency range, then the directions of the propagation vector k and the Poynting vector S in that frequency range are related as

(a)k and S are parallel

(b)k and S are anti-parallel

(c)k and S are perpendicular to each other

(d)k and S make an angle that depends on the magnitude of $|\varepsilon|$ and $|\mu|$

Q.5

The space-time dependence of the electric field of a linearly polarized light in free space is given by $\hat{x}E_0\cos(\omega t - kz)$ where E_0, ω and k are the amplitude, the angular frequency and the wavevector, respectively. The time average energy density associated with the electric field is

(a) $\frac{1}{4}\varepsilon_0 E_0^2$ (b) $\frac{1}{2}\varepsilon_0 E_0^2$ (c) $\varepsilon_0 E_0^2$ (d) $2\varepsilon_0 E_0^2$

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 E^2 \cos^2(wt - kz) \Rightarrow < u_E >= \frac{1}{4}\varepsilon_0 E_0^2$$

Q.6

A plane polarized electromagnetic wave in free space at time $t=0$ is given by $\vec{E}(x,z)=10\hat{j}\exp\left[i(6x+8z)\right]$. The magnetic field $\vec{B}(x,z,t)$ is given by

(a) $\vec{B}(x,z,t)=\frac{1}{c}\left(6\hat{k}-8\hat{i}\right)\exp[i(6x+8z-10ct)]$

(b) $\vec{B}(x,z,t)=\frac{1}{c}\left(6\hat{k}+8\hat{i}\right)\exp[i(6x+8z-10ct)]$

(c) $\vec{B}(x,z,t)=\frac{1}{c}\left(6\hat{k}-8\hat{i}\right)\exp[i(6x+8z-ct)]$

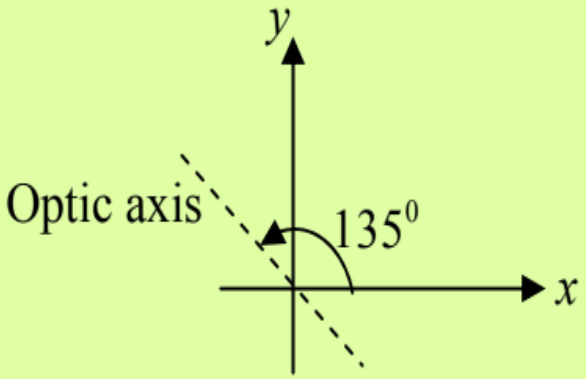
(d) $\vec{B}(x,z,t)=\frac{1}{c}\left(6\hat{k}+8\hat{i}\right)\exp[i(6x+8z+ct)]$

Solution: $\vec{B}=\frac{1}{c}\left(\hat{k}\times\vec{E}\right)=\frac{1}{c}\left(\frac{\vec{k}}{|\vec{k}|}\times\vec{E}\right)=\frac{1}{c}\left(\frac{6\hat{i}+8\hat{k}}{10}\right)\times 10\,\hat{j}\exp\left[i\left(\vec{k}.\vec{r}-\omega t\right)\right]$

$$\vec{B}=\frac{1}{c}\left(6\hat{k}-8\hat{i}\right)\exp[i(6x+8z-10ct)],\quad \omega=10c.$$

Q.7

A quarter wave plate introduces a path difference of $\lambda/4$ between the two components of polarization parallel and perpendicular to the optic axis. An electromagnetic wave with $\vec{E} = (\hat{x} + \hat{y}) E_0 e^{i(kz-\omega t)}$ is incident normally on a quarter wave plate which has its optic axis making an angle 135^0 with the x - axis as shown.



The emergent electromagnetic wave would be

(a) elliptically polarized

(b) circularly polarized

(c) linearly polarized with polarization as that of incident wave

(d) linearly polarized but with polarization at 90^0 to that of the incident wave

Q.8. If the electric and magnetic field in a medium are given by $\tilde{E} = 3 \sin(t - 5z)\,\hat{x}$ and $\tilde{H} = 4 \cos(t - 5z)\,\hat{y}$, then calculate (at z = 0) the

a) instantaneous power density,

b) Instantaneous power transmitted through a surface with an area of 5 m2 at z = 0 and the normal pointing in $\hat{z}$ direction, and

c) total energy carried by the wave through the given surface from t = 0 s to t = 5 s.